Isoperimetric functions for subdirect products and Bestvina-Brady groups

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Abstract

In this thesis we investigate the Dehn functions of two different classes of groups: subdirect products, in particular subdirect products of limit groups; and Bestvina-Brady groups.

Let $D = \Gamma_1 \times \ldots \times \Gamma_n$ be a direct product of $n \geq 3$ finitely presented groups and let H be a subgroup of D. Suppose that each Γ_i contains a finite index subgroup $\Gamma_i' \leq \Gamma_i$ such that the commutator subgroup $[D', D']$ of $D' = \Gamma_1' \times \ldots \times \Gamma_n'$ is contained in H. Suppose furthermore that, for each i, the subgroup $\Gamma_i H$ has finite index in D. We prove that H is finitely presented and satisfies an isoperimetric inequality given in terms of area-radius pairs for the Γ_i and the dimension of $(D'/H) \otimes \mathbb{Q}$. In the case that each Γ_i admits a polynomial-polynomial area-radius pair, it will follow that H satisfies a polynomial isoperimetric inequality.

As a corollary we obtain that if K is a subgroup of a direct product of n limit groups and if K is of type $\mathrm{FP}_m(\mathbb{Q})$, where $m = \max\{2, n-1\}$, then K is finitely presented and satisfies a polynomial isoperimetric inequality. In particular, we obtain that all finitely presented subgroups of a direct product of at most 3 limit groups satisfy a polynomial isoperimetric inequality.

We also prove that if B is a finitely presented Bestvina-Brady group, then B admits a quartic isoperimetric function.

Contents

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1 Introduction

Since its articulation by Dehn in the early 20$^{\text{th}}$ century, the word problem has been one of the guiding problems in combinatorial and geometric group theory. Given some finite group presentation, it asks whether there is an algorithm which will effectively determine whether any given word is trivial in the group. Once it has been determined that a particular group, or class of groups, in which one is interested has a solvable word problem, then it is natural to inquire into the complexity of such an algorithmic solution. In this thesis we study a particular measure of the complexity of the word problem of a group, known as the *Dehn function*.

We give a formal definition in Section 3 below, but, roughly, the Dehn function of a finitely presented group is the least upper bound on the number of defining relations which must be applied to demonstrate that a word in the generators is trivial in the group, with the bound being given in terms of the length of the word. An *isoperimetric function* for a group is an upper bound on the Dehn function. In this thesis we will frequently be concerned with whether a group admits a polynomial isoperimetric function. If one is interested in a class of groups, one might refine this criterion by asking for a single (uniform) polynomial which is an isoperimetric function for all the groups in the class. Some justification for the choice of this dichotomy is provided by a result of Birget, Rips and Sapir [11], who proved that the word problem of a finitely generated group G is an NP-problem if and only if G embeds in a finitely presented group which admits a polynomial isoperimetric function.

Thus far we have discussed Dehn functions in the language of combinatorial group theory. The following geometric interpretation provides further justification for their study. Given a Riemannian manifold M, Plateau's problem asks whether every simple null-homotopic loop in M spans a least-area filling disc. Under mild hypotheses Plateau's problem can be shown to have a positive solution [37], [25], [35], and in this case one can define the *filling function* of M. This is the least function which bounds the area of least-area filling discs of rectifiable null-homotopic loops, with the bound being given in terms of the length of the loop. Gromov's *Filling Theorem* asserts that if M is closed, then its filling function is essentially the same as the Dehn function of $\pi_1 M$.

We now introduce a method for constructing interesting classes of groups that will form the principle objects of study for much of this thesis. Given a class of groups $\mathcal{C}$, the collection of *subdirect products* of $\mathcal{C}$ is defined to be

$$\mathrm{SDP}(\mathcal{C}) = \{S \leq C_1 \times \ldots \times C_n : C_i \in \mathcal{C} \text{ and } S \text{ projects onto each } C_i\}.$$

In many cases the requirement that the subgroup projects onto each factor will be immaterial since one can replace the direct product $C_1 \times \ldots \times C_n$ by $p_1(S) \times \ldots \times p_n(S)$, where $p_i : S \to C_i$ is the projection homomorphism.

Recently, subdirect products have been recognised as worthy objects of study in their own right (see, for example, [13]). Typically, one chooses an input class $\mathcal{C}$ which is already well understood, and asks what can be said about $\mathrm{SDP}(\mathcal{C})$. What is surprising, and fascinating, about this construction is that it only involves two absolutely basic group theoretic operations (taking direct products and passing to subgroups), and yet even when the input class is well understood, the same is not necessarily true of the output class. For example, suppose one takes as input the class $\mathcal{F}$ of free groups: despite this being perhaps the most

basic class of infinite groups, a whole raft of results indicate that the groups in $\mathrm{SDP}(\mathcal{F})$ are surprisingly diverse. Stallings [41] constructed a subgroup of $F \times F \times F$, where F is a rank-2 free group, as the first example of a finitely presented group whose third integral homology group is not finitely generated. Bieri [8] showed that Stallings' group is one element of a sequence of groups $\mathrm{SB}_n \leq F^n$, with SB_n being of type F_{n-1} but not of type FP_n. Baumslag and Roseblade [6] proved that there exist uncountably many finitely generated non-isomorphic subgroups of $F \times F$, and Mihaĭlova [33] and Miller [34] exhibited examples with unsolvable conjugacy problems and unsolvable membership problems. In [13] Bridson and Miller proved that there exists a recursive sequence of finitely generated subgroups $G_i \leq F \times F$ such that there is no algorithm to determine the rank of $H_1(G_i, \mathbb{Z})$, nor to decide whether it has any non-trivial torsion elements.

Hopefully, these examples will have convinced the reader of the inherent wildness of $\mathrm{SDP}(\mathcal{F})$. From our point of view, it is then natural to ask whether this wildness manifests itself in the Dehn functions of these groups.

Question 1. *Does every finitely presented group in* $\mathrm{SDP}(\mathcal{F})$ *admit a polynomial isoperimetric function? Does there exist a uniform polynomial isoperimetric function for the whole class?*

Various authors have obtained results that bear on this question. Gersten [27] proved that, for $n \geq 3$, the Stallings-Bieri group SB_n admits a polynomial isoperimetric function. Elder, Riley, Young and the present author have proved [24] that the Dehn function of Stallings' group SB_3 is actually quadratic. It follows from a theorem of Baumslag and Roseblade (see below) that all of the finitely presented subgroups of a direct product of at most 2 free groups have either linear or quadratic Dehn functions. By a result of Bridson, Howie, Miller and Short (Theorem 1.1 below), the same is true of a subgroup of a direct product of n free groups which satisfies the finiteness condition FP_n. We also note that there are various other lines for investigation naturally related to Question 1. For example, can one find 'nice' presentations for particular groups in $\mathrm{SDP}(\mathcal{F})$? Do there exist finitely presented groups in $\mathrm{SDP}(\mathcal{F})$ whose Dehn functions are actually different from that of the ambient direct product?

Recent results suggest that the wildness encountered amongst the arbitrary finitely generated groups in $\mathrm{SDP}(\mathcal{F})$ is a manifestation of their failure to possess a strong enough degree of finiteness. Baumslag and Roseblade [6] showed that the only finitely presented subgroups of a direct product of 2 free groups are the 'obvious' ones, *i.e.* those which are themselves virtually a direct product of at most 2 free groups. The following result of Bridson, Howie, Miller and Short extends the Baumslag-Roseblade theorem to an arbitrary number of factors.

Theorem 1.1 ([19]). *Let $F_1, \ldots, F_n$ be free groups. A subgroup $G \leq F_1 \times \ldots \times F_n$ is of type FP_n if and only if it has a subgroup of finite index which is itself a direct product of (at most n) free groups.*

Even if a subdirect product does not enjoy any finiteness properties stronger than being finitely presented, one still has the following structural result of Bridson and Miller. Recall that the lower cental series $(\gamma_i(G))_{i=1}^{\infty}$ of a group G is defined recursively by $\gamma_1(G) = G$ and $\gamma_i(G) = [\gamma_{i-1}(G), G]$.

Theorem 1.2 ([13]). *Let $F_1, \ldots, F_n$ be free groups. If a subdirect product $G \leq F_1 \times \ldots \times F_n$ is finitely presented and intersects each factor F_i non-trivially, then each F_i contains a finite index normal subgroup K_i such that*

$$\gamma_{n-1}(K_i) \leq G \cap F_i \leq K_i.$$

In the 3-factor case this yields the following result.

Theorem 1.3 ([13]). *Let F_1, F_2, F_3 be finitely generated free groups and let $G \leq F_1 \times F_2 \times F_3$ be a subdirect product which intersects each factor F_i non-trivially. Then G is finitely presented if and only if each F_i contains a finite index normal subgroup K_i such that the subgroup $G' = G \cap (K_1 \times K_2 \times K_3)$ satisfies the following condition: there is an abelian group Q and epimorphisms $\phi_i : K_i \to Q$ such that G' is the kernel of the map $\phi_1 + \phi_2 + \phi_3$.*

The previous two results suggest that the first step in an attack on Question 1 is to restrict attention to those groups in $\mathrm{SDP}(\mathcal{F})$ which virtually contain the commutator subgroup of the ambient direct product. The BNS invariants (see [9], [10] for definitions) of direct products of free groups have been calculated by Meinert [32] and so, given its finiteness type, one can readily determine how such a co-abelian subgroup sits inside the direct product, and *vice versa*.

One interpretation of Question 1 is as a prototype for a much more profound question regarding the class $\mathcal{L}$ of *limit groups*. In [18] the authors ask the first part of the following question:

Question 2. *Does every finitely presented group in* $\mathrm{SDP}(\mathcal{L})$ *admit a polynomial isoperimetric function? Does there exist a uniform polynomial isoperimetric function for the whole class?*

Limit groups were introduced by Sela ([39] *et seq.*) and separately by Kharalampovich and Myasnikov ([28], [29], [30]) in their solutions to Tarski's question of which groups have the same elementary theory as finitely generated non-abelian free groups. The class contains all finitely generated free and free abelian groups and all compact surface groups of Euler characteristic < -1. In some sense, $\mathcal{L}$ is the class of groups that are 'almost free'; indeed, one fascinating aspect of the theory is that several *a priori* unrelated notions of what it means to be 'almost free' turn out to define the same class of groups.

The simplest definition of limit groups is that they are the finitely generated fully residually free groups, where a group G is defined to be *fully residually free* if for every finite subset $X \subseteq G$ there exists a homomorphism $G \to F$ to a non-abelian free group that is injective on X. From a logical perspective, limit groups are precisely the finitely generated groups with the same existential theory as non-abelian free groups; from a geometric perspective, they are the finitely generated groups that have a Cayley graph in which each ball of finite radius is isometric to a ball of the same radius in some Cayley graph of a free group of finite rank. Limit groups can also be defined in an algebraic context as limits of stable homomorphisms to a free group.

Aside from its own intrinsic interest, several results add further weight to Question 2. It follows from a result of Baumslag, Myasnikov and Remeslennikov [5] and of Sela [39], that the finitely presented groups in $\mathrm{SDP}(\mathcal{L})$ are precisely the finitely presented residually free groups. In a more geometric direction, work of Delzant and Gromov [22] implies that an answer to Question 2 would provide

important information about the isoperimetric behaviour of Kähler groups and compact Kähler manifolds.

Bridson, Howie, Miller and Short [18] have proved that the analogues of Theorems 1.1, 1.2 and 1.3 hold with the words 'free groups' replaced by 'limit groups'. Building on this and other structural results in [17], Kouchloukova [31] proved that if G is a subgroup of a direct product $D = L_1 \times \ldots \times L_n$ of limit groups (with certain additional conditions) and if G is of type $\mathrm{FP}_s(\mathbb{Q})$ for some $s \geq 2$, then the projection homomorphism from G to the direct product of any s of the L_i is virtually surjective. It follows that if G is a subgroup of a direct product of $n \geq 3$ limit groups and if G is of type $\mathrm{FP}_{n-1}(\mathbb{Q})$, then G contains a finite index subgroup G' isomorphic to the kernel of a homomorphism $\phi : L_1 \times \ldots \times L_m \to A$ where $L_1, \ldots, L_m$ are limit groups, A is abelian, $m \leq n$, and restriction of θ to each factor L_i is surjective.

One interpretation of a direct product of free groups is as an example (perhaps the canonical example) of a type of group known as a *right-angled Artin group* (RAAG). Much of the interest in RAAGs amongst geometric group theorists stems from the fact that their definition is flexible enough for them to admit interesting subgroups, and yet they possess enough structure (in particular they have finite $K(\pi_1, 1)$-complexes with the structure of non-positively curved cube complexes) to enable the proof of interesting results. For example, Bestvina and Brady [7] defined a collection of subgroups of RAAGs (known as Bestvina-Brady groups — see Section 14 for definitions) in their solution to the old problem of whether the finiteness conditions F_2 and FP_2 are equivalent. They also constructed a Bestvina-Brady group G such that either G is a counterexample to the Eilenberg-Ganea conjecture, or else there exists a counterexample to the Whitehead conjecture.

In general the richness of the subgroup structure of RAAGs suggests that questions about their arbitrary finitely presented subgroups will be hard. It it thus natural to begin by restricting attention to the Bestvina-Brady subgroups.

Question 3. *Do all finitely presented Bestvina-Brady groups admit a polynomial isoperimetric function? Does their exist a uniform polynomial isoperimetric inequality?*

In [12], Brady suggests that the answer to the second part of this question is no: he constructs a sequence $(\Gamma_k)_{k=1}^{\infty}$ of finitely presented Bestvina-Brady groups and claims that the Dehn function of Γ_k is polynomial of degree $k+2$. However, a result in this thesis shows that in fact n^4 is an isoperimetric function for all finitely presented Bestvina-Brady groups, and hence Brady's construction can not be made to work.

Questions 1–3 acted as the guides for much of the research in this thesis; we have obtained partial answers to Questions 1 and 2, and a complete answer to Question 3. The structure of the thesis is as follows. After describing our notation in Section 2, Section 3 gives the required background on Dehn functions and other related filling invariants. All of this material is standard, although some of the terminology is novel. Section 4 then gives a brief introduction to distortion functions: just as the Dehn function gives a particular measure of the complexity of the word problem for a finitely presented group, so the distortion function gives a measure of the complexity of the membership problem for a pair of finitely generated groups $H \leq G$. Again, the material in this section is standard. Although this thesis is primarily concerned with Dehn functions,

when investigating subdirect products our methods will frequently also give analogous results concerning distortion.

From Section 5 onwards all results are original, except where stated. In Sections 5–10 we prove various general results of a preliminary nature, that are then applied in Sections 11–14 in an attack on Questions 1–3. Each section begins with an introduction explaining its contents.

Guided by Theorems 1.2 and 1.3 (and their limit group analogues) we focus in Section 11 on a class of subdirect products which virtually contain the commutator subgroup of the ambient direct product. For definitions of the terms 'virtually-full', 'virtually-coabelian' and 'corank', see Section 11.1

Theorem A. *Let H be a virtually-full, virtually-coabelian subgroup of a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$, with corank r.*

(1) Suppose each Γ_i is finitely generated and $n \geq 2$. Then H is finitely generated and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.

(2) Suppose each Γ_i is finitely presented and $n \geq 3$. Then H is finitely presented.

(3) Suppose each Γ_i is finitely presented and $n \geq 3$. For each i, let (α_i, ρ_i) be an area-radius pair for some finite presentation of Γ_i. Define

$$\alpha(l) = \max(\{l^2\} \cup \{\alpha_i(l) \,:\, 1 \leq i \leq n\})$$

and

$$\rho(l) = \max(\{l\} \cup \{\rho_i(l) \,:\, 1 \leq i \leq n\}).$$

Then $\rho^{2r}\alpha$ is an isoperimetric function for H

(4) Suppose that each Γ_i is finitely presented and that $n \geq \max\{3, 2r\}$. Let β_1 and β_2 be the Dehn functions of some finite presentations of $\Gamma_1 \times \ldots \times \Gamma_{n-r}$ and $\Gamma_{n-r+1} \times \ldots \times \Gamma_n$ respectively. Then the function β defined by

$$\beta(l) = l\beta_1(l^2) + \beta_2(l)$$

is an isoperimetric function for H.

In Section 12 we focus on subgroups of direct products of limit groups, and use Theorem A to prove the following result.

Theorem B. *Let $L_1, \ldots, L_n$ be limit groups and let H be a subgroup of the direct product $D = L_1 \times \ldots \times L_n$. Suppose that H is of type $\mathrm{FP}_m(\mathbb{Q})$, where $m = \max\{2, n-1\}$. Then H is finitely presented and satisfies a polynomial isoperimetric inequality, and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.*

In particular this result applies to all finitely presented subgroups of a direct product of at most 3 limit groups:

Corollary C. *Let H be a finitely presented subgroup of a direct product D of at most 3 limit groups. Then H satisfies a polynomial isoperimetric inequality and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.*

These results provide a partial solution to Questions 1 and 2.

In Section 13 we focus on a class of subdirect products of free groups which have particularly regular structure. This class includes the Stallings-Bieri groups, and also contains what are perhaps the next most simple groups in $\mathrm{SDP}(\mathcal{F})$ which are not already well understood.

Theorem D. *Let F_1, F_2, F_3 be rank 2 free groups and, for each i, let $\theta_i : F_i \to \mathbb{Z}^2$ be the abelianisation homomorphism. Define $\theta : F_1 \times F_2 \times F_3 \to \mathbb{Z}^2$ to be the homomorphism $\theta_1 + \theta_2 + \theta_3$. Then the kernel of θ is finitely presented and has Dehn function δ satisfying $\delta(l) \succeq l^3$.*

This provides the first known example of a group in $\mathrm{SDP}(\mathcal{F})$ that has Dehn function growing faster than that of the ambient direct product. We also derive an explicit finite presentation for this group.

Finally, in Section 14, we prove the following result, which gives a complete solution to Question 3.

Theorem E. *Every finitely presented Bestvina-Brady group has l^4 as an isoperimetric function.*

2 Notation

Given a set $\mathcal{A}$, write $\mathcal{A}^{-1}$ for the set $\{a^{-1} : a \in \mathcal{A}\}$ of formal inverses to the elements of $\mathcal{A}$ and write $\mathcal{A}^{\pm 1}$ for the set $\mathcal{A} \cup \mathcal{A}^{-1}$. Write $\mathcal{A}^{\pm *}$ for the free monoid on $\mathcal{A}^{\pm 1}$ and $\mathrm{Fr}(\mathcal{A})$ for the free group on $\mathcal{A}$. We call the elements of $\mathcal{A}^{\pm 1}$ *letters* and the elements of $\mathcal{A}^{\pm *}$ *words*. Given words $w_1, w_2 \in \mathcal{A}^{\pm *}$, write $w_1 \equiv w_2$ if w_1 and w_2 are equal as elements of $\mathcal{A}^{\pm *}$ and $w_1 \overset{\mathrm{fr}}{=} w_2$ if w_1 and w_2 are equal as elements of $\mathrm{Fr}(\mathcal{A})$. Write $\emptyset$ for the empty word.

Given a word $w = a_1 \dots a_n \in \mathcal{A}^{\pm *}$, write $|w|$ for the length n of w and $\|w\|$ for the length of the free reduction of w, *i.e.* the length of the unique freely reduced word w' with $w \overset{\mathrm{fr}}{=} w'$. Write $w(i)$ for the i^{th} letter a_i of w and $w[i]$ for the i^{th} prefix $a_1 \dots a_i$ of w. If $i > |w|$ then set $w[i] \equiv w$. Write w^{-1} for the inverse word $a_n^{-1} \dots a_1^{-1}$. Given a set of words $\mathcal{S} \subseteq \mathcal{A}^{\pm *}$, write $\mathcal{S}^{-1}$ for the set of inverses $\{s^{-1} : s \in \mathcal{S}\}$ and $\mathcal{S}^{\pm 1}$ for the set $\mathcal{S} \cup \mathcal{S}^{-1}$. Given words $w_1, \dots, w_n \in \mathcal{A}^{\pm *}$, write $\prod_{j=1}^{n} w_i$ for the concatenated word $w_1 \dots w_n$. Given letters $a_1, a_2 \in \mathcal{A}^{\pm 1}$, write $[a_1, a_2]$ for the word $a_1 a_2 a_1^{-1} a_2^{-1} \in \mathcal{A}^{\pm *}$, write $a_1^{a_2}$ for the word $a_2 a_1 a_2^{-1} \in \mathcal{A}^{\pm *}$, and write $a_1^{-a_2}$ as shorthand for $(a_1^{a_2})^{-1} \equiv a_2 a_1^{-1} a_2^{-1}$. If $\mathcal{A}$ is a generating set for a group G, then write $d_{\mathcal{A}}$ for the word metric on G with respect to $\mathcal{A}$.

As well as considering words as being elements of the free monoid on an alphabet, we sometimes, abusing notation, take the viewpoint that words are maps: we consider a word as being a function which assigns to an ordered set $\mathcal{S}$ of fixed, finite cardinality an element of $\mathcal{S}^{\pm *}$. For example, if $\mathcal{S} = \{x, y\}$ and $\mathcal{S}' = \{x', y'\}$, and $w(\mathcal{S}) = xyx$, then $w(\mathcal{S}') = x'y'x'$. More generally, we will also sometimes consider words which take as input an n-tuple of finite ordered sets $\mathcal{S}_1, \dots, \mathcal{S}_n$ and output a word in $(\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n)^{\pm *}$. In this context, by, for example, $w(\mathcal{S}_1, \emptyset)$ we mean the image of $w(\mathcal{S}_1, \mathcal{S}_2)$ under the projection map $(\mathcal{S}_1 \cup \mathcal{S}_2)^{\pm *} \to \mathcal{S}_1^{\pm *}$. It will always be clear from context whether we are using the term 'word' in the sense of being a map w or in the more usual sense of being an evaluation of w on a specific set.

3 Filling functions

Throughout this section $\mathcal{P} = \langle \mathcal{X} \mid \mathcal{R} \rangle$ is a group presentation with $\mathcal{X}$ finite. We introduce the notions of $\mathcal{P}$-expressions, $\mathcal{P}$-sequences, $\mathcal{P}$-pictures and $\mathcal{P}$-van Kampen diagrams which provide means for representing null-homotopies of words in $\mathcal{X}^{\pm *}$. This allow us to define various *filling invariants*, including Dehn functions, isoperimetric functions and area-radius pairs. Aside from some of the terminology, all of the definitions given here are standard, except that we do not make the usual assumption that $\mathcal{R}$ is finite. For a more thorough introduction to these ideas, see, for example, [15], [38], [26] or [36].

3.1 Representing null-homotopies

Definition 3.1. A word $w \in \mathcal{X}^{\pm *}$ is said to be *null-homotopic* over $\mathcal{P}$ if it represents the identity in the group presented by $\mathcal{P}$.

Definition 3.2 ($\mathcal{P}$-expressions)**.** A $\mathcal{P}$*-expression* is a finite sequence $\mathcal{E} = (x_i, r_i)_{i=1}^m$ of elements of $\mathcal{X}^{\pm *} \times \mathcal{R}^{\pm 1}$. The *area* of $\mathcal{E}$, written $\mathrm{Area}(\mathcal{E})$, is defined to be the integer m. The *radius* of $\mathcal{E}$, written $\mathrm{Rad}(\mathcal{E})$, is defined to be $\max\{|x_i| : 1 \leq i \leq m\}$. We allow the empty sequence which is defined to have both zero area and zero radius. We write $\partial\mathcal{E}$ for the word $\prod_{i=1}^m x_i r_i x_i^{-1}$. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{P}$-expressions then we write $\mathcal{E}_1\mathcal{E}_2$ for the $\mathcal{P}$-expression given by concatenating the two sequences. A $\mathcal{P}$-expression for a word $w \in \mathcal{X}^{\pm *}$ is a $\mathcal{P}$-expression $\mathcal{E}$ with $\partial\mathcal{E}$ freely equal to w.

Definition 3.3 ($\mathcal{P}$-sequences)**.** A $\mathcal{P}$*-sequence* is a sequence $\Sigma = (\sigma_i)_{i=0}^m$ of words in $\mathcal{X}^{\pm *}$ where, for each i, the word σ_{i+1} is obtained from σ_i in one of the following ways:

- **Free contraction:** $\sigma_i \equiv uxx^{-1}v$ and $\sigma_{i+1} \equiv uv$, where $u, v \in \mathcal{X}^{\pm *}$ and $x \in \mathcal{X}^{\pm 1}$.
- **Free expansion:** $\sigma_i \equiv uv$ and $\sigma_{i+1} \equiv uxx^{-1}v$, where $u, v \in \mathcal{X}^{\pm *}$ and $x \in \mathcal{X}^{\pm 1}$.
- **Application-of-a-relator move:** $\sigma_i \equiv urv$ and $\sigma_{i+1} \equiv usv$, where $u, v \in \mathcal{X}^{\pm *}$ and rs^{-1} is a cyclic conjugate of a word in $\mathcal{R}^{\pm 1}$.

Such a $\mathcal{P}$-sequence is said to convert the word σ_0 to the word σ_m. A *null* $\mathcal{P}$*-sequence* for a word $w \in \mathcal{X}^{\pm *}$ is a $\mathcal{P}$-sequence converting w to the empty word $\emptyset$. The *area* of a $\mathcal{P}$-sequence $\Sigma = (\sigma_i)_{i=0}^m$, written $\mathrm{Area}(\Sigma)$, is defined to be the number of i for which the transition from σ_i to σ_{i+1} is an application-of-a-relator move. If $\Sigma_1 = (\sigma_i^{(1)})_{i=0}^{m_1}$ and $\Sigma_2 = (\sigma_i^{(2)})_{i=0}^{m_2}$ are $\mathcal{P}$-sequences with $\sigma_{m_1}^{(1)} \equiv \sigma_0^{(2)}$ then we write $\Sigma_1\Sigma_2$ for the $\mathcal{P}$-sequence $(\sigma_0^{(1)}, \ldots, \sigma_{m_1}^{(1)}, \sigma_1^{(2)}, \ldots, \sigma_{m_2}^{(2)})$. Note that $\mathrm{Area}(\Sigma_1\Sigma_2) = \mathrm{Area}(\Sigma_1) + \mathrm{Area}(\Sigma_2)$.

Definition 3.4 ($\mathcal{P}$-pictures)**.** A $\mathcal{P}$-picture $\mathbb{P}$ consists of a closed 2-disc D (the *ambient disc*); a collection of closed 2-discs $D_1, \ldots, D_m$ (the *relator discs*) embedded pairwise disjointly in the interior of D; and a collection of compact, connected, normally orientated 1-manifolds $\alpha_1, \ldots, \alpha_l$ (the *arcs*) embedded pairwise disjointly in $D \setminus \cup_{i=1}^m \mathrm{Int}\, D_i$. The ambient disc D is equipped with a basepoint $b \in \partial D$, and each relator disc D_i is equipped with a basepoint $b_i \in \partial D_i$. We

require that each arc is disjoint from all basepoints, and that the interior of each arc is disjoint from ∂D and disjoint from each D_i. Each relator disc is labelled by an element of $\mathcal{R}^{\pm 1}$ and each arc is labelled by an element of $\mathcal{X}$.

Reading anticlockwise from its basepoint around the boundary of a relator disc or the ambient disc defines a word in $\mathcal{X}^{\pm *}$, where we understand that if we pass an arc labelled x in the direction of its normal orientation then we read x, and if we pass the arc in the opposite direction to its normal orientation we read x^{-1}. We require that the word associated to each relator disc in this way is precisely the element of $\mathcal{R}^{\pm 1}$ labelling the disc.

The *area* of $\mathbb{P}$, written $\operatorname{Area}\mathbb{P}$, is defined to be the number of relator discs. Define the *background* of $\mathbb{P}$ to be

$$\operatorname{Back}\mathbb{P} := D \smallsetminus \Big((\cup_{i=1}^m D_i) \bigcup (\cup_{i=1}^l \alpha_l) \Big).$$

By a *complementary region* of $\mathbb{P}$ we mean a connected component of $\operatorname{Back}\mathbb{P}$. Given points $p, q \in \operatorname{Back}\mathbb{P}$ a *transverse path* from p to q is a path in $D \smallsetminus \cup_{i=1}^m D_i$ with initial point p and terminal point q which intersects each arc α_i transversely and only finitely many times. Define the *intersection number* of such a path to be the number of times it intersects $\cup_{i=1}^l \alpha_i$. Given a complementary region C, define $d(b, C)$ to be the minimum intersection number over all transverse paths from b to a point in C. Define the *radius* of $\mathbb{P}$, written $\operatorname{Rad}\mathbb{P}$, to be the maximum value of $d(b, C)$ over all complementary regions C.

The *boundary label* of $\mathcal{P}$ is defined to be the word in $\mathcal{X}^{\pm *}$ given by reading anticlockwise around ∂D from the basepoint b. A $\mathcal{P}$-picture $\mathbb{P}$ for a word $w \in \mathcal{X}^{\pm *}$ is a $\mathcal{P}$-picture with boundary label w.

In order to give our fourth, and final, means of representing null-homotopies, namely van Kampen diagrams, we require the notion of a *combinatorial CW-complex.*

Definition 3.5. A cellular map between CW-complexes is said to be *combinatorial* if its restriction to each open cell of the domain complex is a homeomorphism onto some open cell of the codomain complex.

The notion of a CW-complex being *combinatorial* is defined by recursion on dimension. By definition every 0-dimensional CW-complex is combinatorial. An n-dimensional CW-complex X is combinatorial if $X^{(n-1)}$ is combinatorial and for each n-cell e_i^n the attaching map $\theta_i^n : \mathbb{S}^{n-1} \to X^{(n-1)}$ is combinatorial for some combinatorial CW-complex structure on $\mathbb{S}^{n-1}$.

Definition 3.6 ($\mathcal{P}$-van Kampen diagrams)**.** A *singular disc diagram* Δ is a finite, planar, contractible combinatorial CW-complex with a specified base vertex $\star$ in its boundary. The *area* of Δ, written $\operatorname{Area}(\Delta)$, is defined to be the number of 2-cells of which Δ is composed. The *boundary cycle* of Δ is the edge loop in Δ which starts at $\star$ and traverses $\partial\Delta$ in the anticlockwise direction. The interior of Δ consists of a number of disjoint open 2-discs, the closures of which are called the *disc components* of Δ.

Each 1-cell of Δ has associated to it two directed edges ϵ_1 and ϵ_2, with $\epsilon_1^{-1} = \epsilon_2$. Let $\operatorname{DEdge}(\Delta)$ be the set of directed edges of Δ. A *labelling* of Δ over a set $\mathcal{S}$ is a map $\lambda : \operatorname{DEdge}(\Delta) \to \mathcal{S}^{\pm 1}$ such that $\lambda(\epsilon^{-1}) = \lambda(\epsilon)^{-1}$. This induces a map from the set of edge paths in Δ to $\mathcal{S}^{\pm *}$. The *boundary label* of Δ is the word in $\mathcal{S}^{\pm *}$ associated to the boundary cycle.

A $\mathcal{P}$*-van Kampen diagram* for a word $w \in \mathcal{X}^{\pm *}$ is a singular disc diagram Δ labelled over $\mathcal{X}$ with boundary label w and such that for each 2-cell c of Δ the anticlockwise edge loop given by the attaching map of c, starting at some vertex in ∂c, is labelled by a word in $\mathcal{R}^{\pm 1}$.

Definition 3.7 (Cayley complexes)**.** The *presentation* 2*-complex* of $\mathcal{P}$ is a combinatorial 2-complex consisting of a single 0-cell; orientated 1-cells in bijective correspondence with $\mathcal{X}$; and 2-cells in bijective correspondence with $\mathcal{R}$. The 2-cell associated to a relator $r \in \mathcal{R}$ has $|r|$ edges and is attached by identifying its boundary circuit with the edge path along which the word r is read.

The *Cayley* 2*-complex* $Cay^2(\mathcal{P})$ of $\mathcal{P}$ is defined to be the universal cover of the presentation 2-complex. The edges of $Cay^2(\mathcal{P})$ inherit labels and orientations from the presentation 2-complex. If G ia the group presented by P then, after choosing a basepoint, the 0-skeleton of $Cay^2(\mathcal{P})$ is identified with G and there is a natural left action of G on $Cay^2(\mathcal{P})$. The *Cayley graph* $Cay^1(G, \mathcal{X})$ of G with respect to $\mathcal{X}$ is defined to be the 1-skeleton of G.

If Δ is $\mathcal{P}$-van Kampen diagram then there is a unique combinatorial basepoint-preserving and label-preserving map $\Delta \to Cay^2(\mathcal{P})$.

3.2 Dehn functions and the areas of words

Definition 3.8 (van Kampen's Lemma)**.** The following are equivalent for a word $w \in \mathcal{X}^{\pm *}$:

- w is null-homotopic;
- there exists a $\mathcal{P}$-expression for w;
- there exists a null $\mathcal{P}$-sequence for w;
- there exists a $\mathcal{P}$-picture for w;
- there exists a $\mathcal{P}$-van Kampen diagram for w.

Furthermore, if w is null-homotopic, then the following integers are equal:

- $\min\{\text{Area}(\mathcal{E}) : \mathcal{E} \text{ a } \mathcal{P}\text{-expression for } w\}$;
- $\min\{\text{Area}(\Sigma) : \Sigma \text{ a null-}\mathcal{P}\text{-sequence for } w\}$;
- $\min\{\text{Area}(\mathbb{P}) : \mathbb{P} \text{ a } \mathcal{P}\text{-picture for } w\}$;
- $\min\{\text{Area}(\Delta) : \Delta \text{ a } \mathcal{P}\text{-van Kampen diagram for } w\}$;

and these all serve to define the *area* of w, written $\text{Area}(w)$. If we wish to emphasise which presentation we are working with we talk of the $\mathcal{P}$*-area* of w, written $\text{Area}_{\mathcal{P}}(w)$.

Definition 3.9. The *Dehn function* of $\mathcal{P}$ is defined to be the function $\delta_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ given by

$$\delta_{\mathcal{P}}(l) = \max\{\text{Area}(w) \ : \ w \in \mathcal{X}^{\pm *} \text{ is null-homotopic and } |w| \leq l\}.$$

Different finite presentations of the same group may have different Dehn functions, but, in a way which we now make precise, the Dehn functions will have the same asymptotic behaviour.

Definition 3.10. Let f, g be functions $\mathbb{N} \to \mathbb{N}$. Write $f \preceq g$ if there exists a constant $C \in \mathbb{N}$ so that $f(l) \leq Cg(Cl + C) + Cl + C$. Write $f \simeq g$ if $f \preceq g$ and $g \preceq f$.

The following lemma is standard, see for example [15].

Lemma 3.11. *Let Q be a finite presentation presenting the same group as $\mathcal{P}$. Then $\delta_{\mathcal{P}} \simeq \delta_Q$.*

Thus, up to $\simeq$-equivalence, it makes sense to talk about *the* Dehn function of a finitely presented group. We emphasise that although we will sometimes make use of infinite presentations as calculatory tools, the Dehn function of a finitely presented group always refers to the Dehn function of some finite presentation of the group.

Definition 3.12. Let G be a finitely presented group. Then a function $\alpha : \mathbb{N} \to \mathbb{N}$ is said to be an *isoperimetric function* for G if $\delta_{\mathcal{P}} \preceq \alpha$ for some (and hence any) choice of finite presentation $\mathcal{P}$ for G. We say that G satisfies a *polynomial isoperimetric inequality* if it has a polynomial as an isoperimetric function.

3.3 $\mathcal{P}$-schemes

In this thesis we will frequently present bounds on the areas of words, and we wish to convey to the reader how these bounds have been derived. For reasons of space and readability we wish to avoid having to display all of the data required to define a particular null-homotopy. Instead we make use of the notion of null $\mathcal{P}$-schemes, which are essentially skeletons of null-homotopies and which provide enough detail to allow the reader to reconstruct a particular null-homotopy and hence a bound on the area of the word in question.

Definition 3.13. A *$\mathcal{P}$-scheme* consists of a finite sequence of words $(\sigma_i)_{i=1}^m$ in $\mathcal{X}^{\pm *}$ and a finite sequence of integers $(\alpha_i)_{i=1}^{m-1}$ such that, for each i, the word $\sigma_i(\sigma_{i+1})^{-1}$ is null-homotopic over $\mathcal{P}$ with area at most α_i. Such a $\mathcal{P}$-scheme is said to convert the word σ_1 to the word σ_m. We frequently display $\mathcal{P}$-schemes in a table, with the i^{th} row containing the word σ_i and the number α_i. Since there is a disparity between the number of terms in the sequences (σ_i) and (α_i), the last row of such a table will consist of just the word σ_m.

A *null $\mathcal{P}$-scheme* for a word $w \in \mathcal{X}^{\pm *}$ is a $\mathcal{P}$-scheme converting w to the empty word. When displaying a null $\mathcal{P}$-scheme in a table we omit the final row, since this does not contain any non-trivial data. Note that if there exists a null $\mathcal{P}$-scheme for a word w, then w is null-homotopic over $\mathcal{P}$ with area at most the sum of the integers α_i.

As an example, suppose that $\mathcal{P}$ is the presentation $\langle x, y \,|\, [x, y] \rangle$ of a rank 2 free abelian group. The following null $\mathcal{P}$-scheme demonstrates that the word $x^2yx^{-1}yxyx^{-2}y^{-3}$ is null-homotopic over $\mathcal{P}$ with area at most 5.

j	σ_j	Area
1	$x^2yx^{-1}yxyx^{-2}y^{-3}$	2
2	$x^2yx^{-1}yx^{-1}y^{-2}$	1
3	$x^2yx^{-2}y^{-1}$	2
	Total	5

3.4 Area-radius pairs

As well as bounding the areas of $\mathcal{P}$-expressions for words, we sometimes wish to simultaneously bound their radii.

Definition 3.14. A pair of functions (α, ρ), each $\mathbb{N} \to \mathbb{N}$, is said to be an *area-radius pair* for $\mathcal{P}$ if, for every null-homotopic word $w \in \mathcal{X}^{\pm *}$ with $|w| \leq l$, there exists a $\mathcal{P}$-expression $\mathcal{E}$ for w with $\mathrm{Area}(\mathcal{E}) \leq \alpha(l)$ and $\mathrm{Rad}(\mathcal{E}) \leq \rho(l)$.

The following result shows how area-radius pairs transform under change of presentation.

Proposition 3.15. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite presentations of the same group. If (α, ρ) is an area-radius pair for $\mathcal{P}$ then there exists an area-radius pair (α', ρ') for $\mathcal{Q}$ with $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$.*

Proof. Since $\mathcal{P}$ can be converted to $\mathcal{Q}$ by a finite sequence of Tietze transformations, it suffices to prove the proposition in the situation that $\mathcal{P}$ and $\mathcal{Q}$ are related by a single such transformation. There are four cases to consider.

Case 1. Suppose that $\mathcal{P} = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ and $\mathcal{Q} = \langle \mathcal{A} \,|\, \mathcal{R}, s \rangle$ where $s \in \mathcal{A}^{\pm *}$ is null-homotopic over $\mathcal{P}$. A $\mathcal{P}$-expression for a word $w \in \mathcal{A}^{\pm *}$ is also a $\mathcal{Q}$-expression for w, so (α, ρ) is itself an area-radius pair for $\mathcal{Q}$.

Case 2. Suppose that $\mathcal{P} = \langle \mathcal{A} \,|\, \mathcal{R}, s \rangle$ and $\mathcal{Q} = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ where $s \in \mathcal{A}^{\pm *}$ is null-homotopic over $\mathcal{Q}$. Let $(x_i, r_i)_{i=1}^{M}$ be a $\mathcal{Q}$-expression for s with area M and radius K. If $w \in \mathcal{A}^{\pm *}$ is a null-homotopic word of length at most n then there exists a $\mathcal{P}$-expression $\Sigma = (y_i, z_i)_{i=1}^{L}$ for w with area $L \leq \alpha(n)$ and radius at most $\rho(n)$. Substituting $\prod_{i=1}^{M} x_i r_i x_i^{-1}$ for each occurrence of s in the product $\prod_{i=1}^{L} y_i z_i y_i^{-1}$ gives a product which is freely equal to w in $F(\mathcal{A})$. The corresponding $\mathcal{Q}$-expression has area at most ML and radius at most $\rho(n) + K$. Thus $(M\alpha(n), \rho(n) + K)$ is an area-radius pair for $\mathcal{Q}$.

Case 3. Suppose that $\mathcal{P} = \langle \mathcal{A} \,|\mathcal{R} \rangle$ and $\mathcal{Q} = \langle \mathcal{A}, b \,|\, \mathcal{R}, bu_b^{-1} \rangle$ where $u_b \in \mathcal{A}^{\pm *}$ and bu_b^{-1} is null-homotopic over $\mathcal{P}$. Define $K = |u_b|$. Suppose $w \in (\mathcal{A} \cup \{b\})^{\pm *}$ is a null-homotopic word of length at most n; say $w \equiv v_0 b^{\epsilon_1} v_1 \ldots b^{\epsilon_L} v_L$ for some $v_i \in \mathcal{A}^{\pm *}$ and $\epsilon_i \in \{\pm 1\}$. Insert cancelling pairs $u_b^{-1} u_b$ into w to obtain the word $w' \equiv v_0 (bu_b^{-1} u_b)^{\epsilon_1} v_1 \ldots (bu_b^{-1} u_b)^{\epsilon_L} v_L$ with $w' \stackrel{\mathrm{fr}}{=} w$. Define $v'_0, \ldots, v'_L$ to be the words in $\mathcal{A}^{\pm *}$ such that $w' \equiv v'_0 (bu_b^{-1})^{\epsilon_1} v'_1 \ldots (bu_b^{-1})^{\epsilon_L} v'_L$ and note that $\sum_{i=1}^{L} |v'_i| \leq K|w| \leq Kn$. For each $i \in \{0, \ldots, L\}$ define $\tau_i \equiv v'_i v'_{i+1} \ldots v'_L$. Then

$$w' \stackrel{\mathrm{fr}}{=} \tau_0 \prod_{i=1}^{L} \tau_i^{-1} (bu_b)^{\epsilon_i} \tau_i$$

and $|\tau_i| \leq \sum_{i=1}^{L} |v'_i| \leq Kn$. The word τ_0 is null-homotopic over $\mathcal{Q}$ and hence over $\mathcal{P}$ and so there exists a $\mathcal{P}$-expression $(x_i, r_i)_{i=1}^{M}$ for τ_0 with area at most

$\alpha(Kn)$ and radius at most $\rho(Kn)$. Thus

$$w \overset{\text{fr}}{=} \prod_{i=1}^{M} x_i r_i x_i^{-1} \prod_{i=1}^{L} \tau_i^{-1}(bu_b^{-1})^{\epsilon_i}\tau_i$$

and so we obtain a $\mathcal{Q}$-expression for w with area at most $M+L \leq \alpha(Kn)+n$ and radius at most $\max\{\max_i |x_i|, \max_i |v_i'|\} \leq \max\{\rho(Kn), Kn\} \leq \rho(Kn) + Kn$. Thus $(\alpha(Kn)+n, \rho(Kn)+Kn)$ is an area-radius pair for $\mathcal{Q}$.

Case 4. Suppose that $\mathcal{P} = \langle \mathcal{A}, b \,|\, \mathcal{R}, bu_b^{-1}\rangle$ and $\mathcal{Q} = \langle \mathcal{A} \,|\, \mathcal{R}\rangle$ where $u_b \in \mathcal{A}^{\pm *}$ and bu_b^{-1} is null-homotopic over $\mathcal{Q}$. Define $K = |u_b|$. Consider the retraction $\pi : (\mathcal{A} \cup \{b\})^{\pm *} \to \mathcal{A}^{\pm *}$ which is the identity on $\mathcal{A}$ and maps $b^{\pm 1} \mapsto u_b^{\pm 1}$. Note that π induces a retraction $F(\mathcal{A} \cup \{b\}) \to F(\mathcal{A})$. Suppose $w \in \mathcal{A}^{\pm *}$ is a null-homotopic word of length at most n and let $(x_i, z_i)_{i=1}^{M}$ be a $\mathcal{P}$-expression for w with area at most $\alpha(n)$ and radius at most $\rho(n)$. Let S be the subset of $\{1, \ldots, m\}$ consisting of those i for which $z_i \in \mathcal{R}^{\pm 1}$. Then $(\pi(x_i), \pi(z_i))_{i \in S}$ is a $\mathcal{Q}$-expression for w with area at most M and radius at most $K\rho(n)$. Thus $(\alpha(n), K\rho(n))$ is an area-radius pair for $\mathcal{Q}$. $\square$

As with the areas of words, area-radius pairs have interpretations in terms of $\mathcal{P}$-sequence, $\mathcal{P}$-pictures and $\mathcal{P}$-van Kampen diagrams; of these we only consider the pictorial interpretation.

Definition 3.16. A pair (α, ρ) of functions $\alpha, \rho : \mathbb{N} \to \mathbb{N}$ is said to be a *pictorial area-radius pair* for the presentation $\mathcal{P}$ if, for all null-homotopic words $w \in \mathcal{A}^{\pm *}$ with $|w| \leq l$, there exists a $\mathcal{P}$-picture $\mathbb{P}$ for w with $\operatorname{Area} \mathbb{P} \leq \alpha(l)$ and $\operatorname{Rad} \mathbb{P} \leq \rho(l)$.

Proposition 3.17. *If (α, ρ) is an area-radius pair for a presentation $\mathcal{P}$ then there exists a pictorial area-radius pair (α', ρ') for $\mathcal{P}$ with $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$. Conversely if (α, ρ) is a pictorial area-radius pair for $\mathcal{P}$ then there exists an area-radius pair (α', ρ') for $\mathcal{P}$ with $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$.*

The only place in this thesis where we make use of Proposition 3.17 is in the proof of Theorem 7.5. We thus omit the proof of this proposition since Theorem 7.5 is implied by the stronger Theorem 7.4.

3.5 Finite index subgroups

We will frequently simplify arguments by passing to finite index subgroups. The following lemma shows that Dehn functions and area-radius pairs are unaffected by this transition.

Lemma 3.18. *Let $H \leq G$ be a pair groups with finite presentations $\mathcal{P}$ and $\mathcal{Q}$ respectively. Suppose that H has finite index in G.*

(1) Let $\delta_{\mathcal{P}}$ and $\delta_{\mathcal{Q}}$ be the Dehn functions of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Then $\delta_{\mathcal{P}} \simeq \delta_{\mathcal{Q}}$.

(2) Let (α, ρ) be an area-radius pair for $\mathcal{Q}$. Then there exists an area-radius pair (α', ρ') for $\mathcal{P}$ with $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$.

NB: It is also true that if (α, ρ) is an area-radius pair for $\mathcal{P}$ then there exists an area-radius pair (α', ρ') for $\mathcal{Q}$ with $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$. However we will not need this result.

Proof of Lemma 3.18. For (1), observe that since H has finite index in G, these two groups are quasi-isometric. The result then follows since quasi-isometric groups have $\simeq$-equivalent Dehn functions [2].

The assertion (2) is standard. We give our own proof in Section 10 as a corollary to Proposition 10.4. $\square$

4 Distortion Functions

Let $H \leq G$ be a pair of groups with finite generating sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. The *distortion function* of H in G with respect to $\mathcal{X}$ and $\mathcal{Y}$ is defined to be the function $\Delta : \mathbb{N} \to \mathbb{N}$ given by

$$\Delta(l) = \max\{d_{\mathcal{X}}(1, h) \,:\, h \in H, d_{\mathcal{Y}}(1, h) \leq l\}.$$

Different choices of generating sets will give rise to different distortion functions, but, in a way we now make precise, these will have the same asymptotic behaviour.

Definition 4.1. Let f, g be functions $\mathbb{N} \to \mathbb{N}$. Write $f \preccurlyeq g$ if there exists a constant $C \in \mathbb{N}$ so that $f(l) \leq Cg(Cl)$. Write $f \approx g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

The following lemma is standard.

Lemma 4.2. *For each $i = 1, 2$, let Δ_i be the distortion function of H in G with respect to some finite generating sets $\mathcal{X}_i$ and $\mathcal{Y}_i$ for H and G respectively. Then $\Delta_1 \approx \Delta_2$.*

Thus we may talk of *the* distortion function of H in G, without making any mention of a choice of generating sets, provided we bear in mind that this is only defined up to $\approx$-equivalence.

We say H has *polynomial distortion* in G if the distortion function with respect to some (and hence any) finite generating sets is bounded above by a polynomial. We say H is *undistorted* in G if the distortion function with respect to some (and hence any) finite generating sets is linear. For example, finite index subgroups are undistorted, as are direct factors or, more generally, retracts.

The following lemma gives various transitivity properties of distortion functions.

Lemma 4.3. *Let $G_1 \leq G_2 \leq G_3$ be groups with finite generating sets $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ respectively. For each $1 \leq i < j \leq 3$, let Δ_i^j be the distortion function of G_i in G_j with respect to $\mathcal{X}_i$ and $\mathcal{X}_j$.*

(1) $\Delta_1^3(l) \leq \Delta_1^2(\Delta_2^3(l))$.

(2) If G_1 has finite index in G_2 then $\Delta_2^3 \approx \Delta_1^3$.

(3) If G_2 has finite index in G_3 then $\Delta_1^2 \approx \Delta_1^3$.

Proof. Property (1) is immediate.

For (2), the direction $\Delta_1^3 \preccurlyeq \Delta_2^3$ follows immediately from property (1). For the converse, note that by Lemma 4.2 we are at liberty to choose any finite generating sets convenient to our purposes. Choose $\mathcal{X}_2$ to contain a collection $k_1, \ldots, k_n$ of right coset representatives of G_1 in G_2 and choose $\mathcal{X}_3$ to contain $\mathcal{X}_2$.

Let $w \in \mathcal{X}_3^{\pm *}$ represent a non-identity element of G_2. Then there exists i so that wk_i^{-1} represents an element of G_1. Since $wk_i^{-1} \in \mathcal{X}_3^{\pm *}$, there exists $w' \in \mathcal{X}_1^{\pm *}$ representing the same element as wk_i^{-1} with $|w'| \leq \Delta_1^3(|w|+1) \leq \Delta_1^3(2|w|)$. Then $w'k_i \in \mathcal{X}_2^{\pm *}$ represents w and has length at most $\Delta_1^3(2|w|)+1 \leq 2\Delta_1^3(2|w|)$.

For (3), the direction $\Delta_1^3 \preccurlyeq \Delta_1^2$ follows immediately from property (1). For the converse, choose the generating set $\mathcal{X}_3$ to contain $\mathcal{X}_2$. Then $\mathcal{X}_2^{\pm *} \subseteq \mathcal{X}_3^{\pm *}$ and so $\Delta_1^2(l) \leq \Delta_1^3(l)$ for all l. $\square$

Corollary 4.4. *Let H, H' and G' be finitely generated subgroups of the finitely generated group G, with $H' \leq H \cap G'$. Suppose that H' has finite index in H and G' has finite index in G. Let Δ and Δ' be the distortion functions of H in G and H' in G' respectively. Then $\Delta \approx \Delta'$.*

Lemma 4.5. *Let $H \leq G$ be finitely generated groups and let $p : G \to G'$ be a surjective homomorphism which is injective on H. Let Δ and Δ' be the distortion functions of H in G and $p(H)$ in G' respectively. Then $\Delta \preccurlyeq \Delta'$.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite generating sets for H and G respectively. Define $\mathcal{X}' = p(\mathcal{X})$ and $\mathcal{Y}' = p(\mathcal{Y})$ and note that these are finite generating sets for $p(H)$ and G' respectively. By Lemma 4.2 we may assume that Δ and Δ' are defined with respect to these generating sets. Then, for any $g_1, g_2 \in G$ one then has that $\mathrm{d}_{\mathcal{Y}'}(p(g_1), p(g_2)) \leq \mathrm{d}_{\mathcal{Y}}(g_1, g_2)$. Since the restriction of p to H is an isomorphism onto its image, $\mathrm{d}_{\mathcal{X}'}(p(h_1), p(h_2)) = \mathrm{d}_{\mathcal{X}}(h_1, h_2)$ for all $h_1, h_2 \in H$. Thus for any $l \in \mathbb{N}$, we have the inclusion of sets

$$\{\mathrm{d}_{\mathcal{X}}(1,h) : h \in H, \mathrm{d}_{\mathcal{Y}}(1,h) \leq l\} \subseteq \{\mathrm{d}_{\mathcal{X}'}(1,h) : h \in p(H), \mathrm{d}_{\mathcal{Y}'}(1,h) \leq l\}.$$

It follows that $\Delta(l) \leq \Delta'(l)$. $\square$

5 The Bounded Noise Lemma

Let $\mathcal{P} = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ be a finite presentation with area-radius pair (α, ρ) and define $L = \max\{|r| : r \in \mathcal{R}\}$. If w is a null-homotopic word over $\mathcal{P}$ with $|w| \leq n$ then there exists a $\mathcal{P}$-expression $\mathcal{E}$ for w with area $\leq \alpha(n)$ and radius $\leq \rho(n)$. Thus $|\partial\mathcal{E}| \leq (2\rho(n)+L)\alpha(n)$. The Bounded Noise Lemma shows that $\mathcal{E}$ can be chosen so that the free reduction of the word $\partial\mathcal{E}$ is bounded only in terms of α. This lemma is not original, but a proof of it does not appear to exist in the literature. Recall that we write $|w|$ for the length of a word w, and $\|w\|$ for the length of the free reduction of w.

Lemma 5.1 (The Bounded Noise Lemma)**.** *Let w be a null-homotopic word over the presentation $\mathcal{P}$ with area N. Then there exists a $\mathcal{P}$-expression $(u_i, r_i)_{i=1}^N$ for w with*

$$\|u_1\| + \sum_{i=1}^{N-1} \|u_i^{-1}u_{i+1}\| + \|u_N\| \leq |w| + 2LN.$$

The proof of this result makes use of the following notions concerning van Kampen diagrams. Say the anticlockwise boundary cycle of a van Kampen diagram D, read from the base vertex, is given by the edge path $e_1 \cdot \ldots \cdot e_k$, where $e_1, \ldots, e_k$ are edges of D (possibly with repetition) and $\cdot$ denotes concatenation. Let e_i be the first edge lying in the boundary of some 2-cell of D. Then we call

the edge e_i the *first thick boundary edge* of D and the edge path $e_1 \cdot \ldots \cdot e_{i-1}$ the *initial boundary segment* of D.

Proof. We actually prove the following:

Claim. *Let Δ be a $\mathcal{P}$-van Kampen diagram for the word w with area N. Then there exist words $s_1, \ldots, s_N \in \mathcal{A}^{\pm *}$ labelling 2-cells of Δ, each read anticlockwise from some vertex, and there exist words $v_1, \ldots, v_N \in \mathcal{A}^{\pm *}$ with v_1 the label on the initial boundary segment of Δ, such that $(v_i, s_i)_{i=1}^N$ is a $\mathcal{P}$-expression for w and*

$$|v_1| + \sum_{i=1}^{N-1} \|v_i^{-1} v_{i+1}\| + \|v_N\| \leq |w| + LN.$$

The lemma as stated follows from the claim since each s_i is a cyclic conjugate of some relator $r_i \in \mathcal{R}^{\pm 1}$ and so is freely equal to a word $x_i r_i x_i^{-1}$ for some $x_i \in \mathcal{A}^{\pm *}$ with $|x_i| \leq |r_i|/2 \leq L/2$. It follows that we can set $u_i = v_i x_i$ and then $(u_i, r_i)_{i=1}^N$ is a $\mathcal{P}$-expression for w and

$$\begin{aligned}
&\|u_1\| + \sum_{i=1}^{N-1} \|u_i^{-1} u_{i+1}\| + \|u_N\| \\
&\quad = \|v_1 x_1\| + \sum_{i=1}^{N-1} \|x_i^{-1} v_i^{-1} v_{i+1} x_{i+1}\| + \|v_N x_N\| \\
&\quad \leq \|v_1\| + \|x_1\| + \sum_{i=1}^{N-1} \left(\|v_i^{-1} v_{i+1}\| + \|x_i\| + \|x_{i+1}\|\right) + \|v_N\| + \|x_N\| \\
&\quad \leq |v_1| + |x_1| + \sum_{i=1}^{N-1} \left(\|v_i^{-1} v_{i+1}\| + |x_i| + |x_{i+1}|\right) + \|v_N\| + |x_N| \\
&\quad \leq |w| + LN + L/2 + L/2 + (N-1)(L/2 + L/2) \\
&\quad = |w| + 2LN.
\end{aligned}$$

The claim is proved by induction on the area of Δ. If Δ has area 0 the conclusion is trivial. Now suppose that Δ has area $N \geq 1$ and that the claim is true for diagrams with smaller area. Say Δ has boundary label w and initial boundary segment labelled by the word v_1. Let e be the first thick boundary edge of Δ and let c be the unique 2-cell of Δ that contains e in its boundary. The anticlockwise orientation of the boundary cycle of Δ induces an orientation on the edge e. Say c has boundary label s_1 read anticlockwise from the origin of e.

Let Δ' be the van Kampen diagram of area $N-1$ formed from Δ by deleting the (interior of the) 2-cell c and the (interior of the) edge e. Say Δ' has boundary label w'. Observe that w is freely equal to the word $v_1 s_1 v_1^{-1} w'$ and that $|w'| \leq |w| + L$. Applying the induction hypothesis to Δ' gives that there exist $v_2, \ldots, v_N \in \mathcal{A}^{\pm *}$ with v_2 the label on the initial boundary segment of Δ' and there exist $s_2, \ldots, s_N \in \mathcal{A}^{\pm *}$ labelling 2-cells of Δ' such that

$$w' \overset{\text{fr}}{=} \prod_{i=2}^{N} v_i s_i v_i^{-1}$$

and

$$|v_2| + \sum_{i=2}^{N-1} \|v_i^{-1}v_{i+1}\| + \|v_N\| \leq |w'| + L(N-1).$$

Thus

$$w \stackrel{\text{fr}}{=} \prod_{i=1}^{N} v_i s_i v_i^{-1}.$$

By construction the initial boundary segment of Δ' is formed by concatenating the initial boundary segment of Δ with a (possibly empty) edge path γ. Let α be the label on γ, so $v_2 \equiv v_1\alpha$. Then $|v_2| = |v_1|+|\alpha| \geq |v_1|+\|\alpha\| = |v_1|+\|v_1^{-1}v_2\|$ and so

$$\begin{aligned}
|v_1| + \sum_{i=1}^{N-1} \|v_i^{-1}v_{i+1}\| + \|v_N\| &\leq |v_2| - \|v_1^{-1}v_2\| + \sum_{i=1}^{N-1} \|v_i^{-1}v_{i+1}\| + \|v_N\| \\
&= |v_2| + \sum_{i=2}^{N-1} \|v_i^{-1}v_{i+1}\| + \|v_N\| \\
&\leq |w'| + L(N-1) \\
&\leq |w| + LN.
\end{aligned}$$

□

6 Infinite presentations

In the process of deriving a finite presentation for a group, we will sometimes find it useful to first produce, as an intermediate stage, a presentation with infinitely many relations. Care must be taken when dealing with the isoperimetry of such non-finite presentations. The Dehn functions of different finite presentations of a fixed group all have the same asymptotic behaviour. However, the same is not true for presentations with an infinite number of relators, where the behaviour of the Dehn functions may differ markedly. Indeed, for any group, if we take the set of relators to consist of all null-homotopic words then we obtain a presentation whose Dehn function is constant. In order to regain some control over how the Dehn function changes when changing between (possibly non-finite) presentations, we introduce the following notions.

Definition 6.1. An *index* on a set $\mathcal{X}$ is a function $\|\cdot\| : \mathcal{X} \to \mathbb{N}$. This is extended to an index on the set $\mathcal{X}^{\pm 1}$ by setting $\|x^{-1}\| = \|x\|$. An *indexed presentation* is a pair $(\mathcal{P}, \|\cdot\|)$ where $\mathcal{P} = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ is a presentation and $\|\cdot\|$ is an index on $\mathcal{R}$.

Let $(\mathcal{P}, \|\cdot\|)$ be an indexed presentation whose set of generators $\mathcal{A}$ is finite. A pair (α, π) of functions $\alpha, \pi : \mathbb{N} \to \mathbb{N}$ is said to be an *area-penetration pair* for $(\mathcal{P}, \|\cdot\|)$ if for all null-homotopic words $w \in \mathcal{A}^{\pm *}$ with $|w| \leq n$ there exists a null-$\mathcal{P}$-expression $(x_i, r_i)_{i=1}^m$ for w with area $m \leq \alpha(n)$ and with $\|r_i\| \leq \pi(n)$ for each i.

Let $\mathcal{Q} = \langle \mathcal{A} \,|\, \mathcal{S} \rangle$ be a presentation with each $s \in \mathcal{S}$ null-homotopic over $\mathcal{P}$ and each $r \in \mathcal{R}$ null-homotopic over $\mathcal{Q}$. Thus $\mathcal{P}$ and $\mathcal{Q}$ present the same group.

The *relational area function* of $(\mathcal{P}, \|\cdot\|)$ over $\mathcal{Q}$ is defined to be the function $\mathbb{N} \to \mathbb{N} \cup \{\infty\}$ given by

$$\mathrm{RArea}(n) = \max\{\mathrm{Area}_{\mathcal{Q}}(r) \,:\, r \in \mathcal{R}, \|r\| \leq n\}.$$

Proposition 6.2. *Let $(\mathcal{P}, \|\cdot\|)$ and $\mathcal{Q}$ be as in definition 6.1. Let (α, π) be an area-penetration pair for $(\mathcal{P}, \|\cdot\|)$ and let* RArea *be the relational area function of $(\mathcal{P}, \|\cdot\|)$ over $\mathcal{Q}$. Then the Dehn function $\delta_{\mathcal{Q}}$ of the presentation $\mathcal{Q}$ satisfies*

$$\delta_{\mathcal{Q}}(n) \leq \alpha(n)\,\mathrm{RArea}(\pi(n)).$$

Proof. Let $w \in \mathcal{A}^{\pm *}$ be a null-homotopic word with $|w| \leq n$. Then there exist $\sigma_1, \ldots, \sigma_N \in \mathcal{A}^{\pm *}$ and $r_1, \ldots, r_N \in \mathcal{R}^{\pm 1}$ with $N \leq \alpha(n)$ and $\|r_i\| \leq \pi(n)$ for each i such that

$$w \stackrel{\mathrm{fr}}{=} \prod_{i=1}^{N} \sigma_i r_i \sigma_i^{-1}.$$

For each i we have that $\mathrm{Area}_{\mathcal{Q}}(r_i) \leq \mathrm{RArea}(\|r_i\|) \leq \mathrm{RArea}(\pi(n))$ and therefore there exist $\tau_{i1}, \ldots, \tau_{iM_i} \in \mathcal{A}^{\pm *}$ and $s_{i1}, \ldots, s_{iM_i} \in \mathcal{S}^{\pm 1}$ with $M_i \leq \mathrm{RArea}(\pi(n))$ such that

$$r_i \stackrel{\mathrm{fr}}{=} \prod_{j=1}^{M_i} \tau_{ij} s_{ij} \tau_{ij}^{-1}.$$

Hence

$$w \stackrel{\mathrm{fr}}{=} \prod_{i=1}^{N} \prod_{j=1}^{M_i} (\sigma_i \tau_{ij}) s_{ij} (\sigma_i \tau_{ij})^{-1}$$

and so $\mathrm{Area}_{\mathcal{Q}}(w) \leq \sum_{i=1}^{N} M_i \leq \alpha(n)\ \mathrm{RArea}(\pi(n))$. ☐

Section 7 contains a result, Theorem 7.4, concerning area-penetration pairs and cyclic extensions. Although we give a full algebraic proof of this theorem, the intuition behind it derives from the pictorial context and so we will sketch a proof of the slightly weaker Theorem 7.5 in this language. We will thus need the pictorial analogue of area-penetration pairs.

Definition 6.3. Let $(\mathcal{P}, \|\cdot\|)$ be an indexed presentation whose set of generators $\mathcal{A}$ is finite. A pair (α, π) of functions $\alpha, \pi : \mathbb{N} \to \mathbb{N}$ is said to be a *pictorial area-penetration pair* for $(\mathcal{P}, \|\cdot\|)$ if for all null-homotopic words $w \in \mathcal{A}^{\pm *}$ with $|w| \leq n$ there exists a picture $\mathbb{P}$ with boundary label w such that $\mathrm{Area}\,\mathbb{P} \leq \alpha(n)$ and $\|r\| \leq \pi(n)$ for each relator r of $\mathcal{P}$ labelling a relator disc of $\mathbb{P}$.

Proposition 6.4. *A pair (α, π) of functions $\alpha, \pi : \mathbb{N} \to \mathbb{N}$ is an area-penetration pair for a presentation $\mathcal{P}$ if and only if it is a pictorial area-penetration pair for $\mathcal{P}$.*

Since we do not rely on this proposition for the proof of Theorem 7.4, we omit its proof.

7 Cyclic extensions

Let $1 \to K \to \Gamma \to \mathbb{Z} \to 1$ be a cyclic extension with K (and hence Γ) finitely generated. In all of the applications presented in this thesis, Γ will be finitely presented, but we do not need to make this assumption. In the principal result of this section (Theorem 7.4) we show how a presentation $\mathcal{P}_\Gamma$ of Γ (of a certain form) gives rise to an infinite presentation $\mathcal{P}_K^\infty$ for K. The relators of $\mathcal{P}_K^\infty$ come equipped with an index $\|\cdot\|$ and we prove that an area-radius pair for $\mathcal{P}_\Gamma$ is actually an area-penetration pair for $(\mathcal{P}_K^\infty, \|\cdot\|)$. However, before we introduce this new material, we first recall a result of Baik-Harlander-Pride.

Let $\mathcal{A}$ be a finite generating set for K and let $t \in \Gamma$ be an element whose image generates $\Gamma/K \cong \mathbb{Z}$. Let θ be the automorphism of K induced by conjugation by t. For each $a \in \mathcal{A}$ and $\epsilon \in \{\pm 1\}$, let $w_a^\epsilon \in \mathcal{A}^{\pm *}$ be a word representing $t^\epsilon a t^{-\epsilon}$ in K. For each $\epsilon \in \{\pm 1\}$, define $\mathcal{S}^\epsilon = \{t^\epsilon a t^{-\epsilon}(w_a^\epsilon)^{-1} : a \in \mathcal{A}\}$. Furthermore, define an endomorphism $\Phi^\epsilon : \mathcal{A}^{\pm *} \to \mathcal{A}^{\pm *}$, commuting with the inversion automorphism, by mapping $a \mapsto w_a^\epsilon$.

Theorem 7.1 (Baik-Harlander-Pride [3, Theorem 6.1])**.** *Let $\langle \mathcal{A}, t \,|\, \mathcal{R}, \mathcal{S}^+, \mathcal{S}^- \rangle$ be a presentation for Γ with $\mathcal{R} \subseteq \mathcal{A}^{\pm *}$. Suppose that all the relations in the sets $\{a\Phi^-(\Phi^+(a))^{-1} : a \in \mathcal{A}\}$ and $\{\Phi^\epsilon(r) : \epsilon \in \{\pm 1\}, r \in \mathcal{R}\}$ are null-homotopic over the presentation $\langle \mathcal{A} \,|\, \mathcal{R} \rangle$. Then K is presented by $\langle \mathcal{A} \,|\, \mathcal{R} \rangle$.*

We will apply Theorem 7.1 in Section 13 to derive finite presentations for certain subdirect products of free groups. However, the proof of this result in [3] is based on successively removing t-rings from van Kampen diagrams over the presentation $\langle \mathcal{A}, t \,|\, \mathcal{R}, \mathcal{S}^+, \mathcal{S}^- \rangle$, a method which will in general only give an exponential isoperimetric function for K. Since we will be interested in producing polynomial isoperimetric inequalities we adopt a different approach, which essentially involves removing all t-rings simultaneously. We begin with a minor technicality.

Definition 7.2. A presentation $\langle \mathcal{A}, t \,|\, \mathcal{T} \rangle$ for Γ is said to be in *positive normal form* if, for each $a \in \mathcal{A}$, there is precisely one relator in $\mathcal{T}$ of the form $tat^{-1}w$ with $w \in \mathcal{A}^{\pm *}$, *and*, all the relators in $\mathcal{T}$ involving t are of this form.

Thus, given words w_a^+ as defined above, a presentation $\langle \mathcal{A}, t \,|\, \mathcal{R}, \mathcal{S}^+ \rangle$ for Γ with $\mathcal{R} \subseteq \mathcal{A}^{\pm *}$ is in positive normal form. In particular, if $\langle \mathcal{A} \,|\, \mathcal{R} \rangle$ is a presentation for K, then $\langle \mathcal{A}, t \,|\, \mathcal{R}, \mathcal{S}^+ \rangle$ is in positive normal form. The following lemma shows that restricting our attention to positive normal form presentations does not impinge on the generality of our results.

Lemma 7.3. *If Γ is finitely presented then it is presented by some finite presentation in positive normal form.*

Proof. Let $\langle \mathcal{A} | \mathcal{R} \rangle$ be an arbitrary (not necessarily finite) presentation for K. Then Γ is presented by the positive normal form presentation $\langle \mathcal{A}, t \,|\, \mathcal{R}, \mathcal{S}^+ \rangle$. Since Γ is finitely presented there is some finite subcollection of $\mathcal{R} \cup \mathcal{S}^+$ which suffice as a set of defining relators. In particular, there exists a finite subset $\mathcal{R}' \subseteq \mathcal{R}$ so that Γ is finitely presented by $\langle \mathcal{A}, t \,|\, \mathcal{R}', \mathcal{S}^+ \rangle$. ∎

Now let $\mathcal{P}_\Gamma = \langle \mathcal{A}, t \,|\, \mathcal{R}, \mathcal{S} \rangle$ be a positive normal form presentation for Γ with $\mathcal{R} \subseteq \mathcal{A}^{\pm *}$ and $\mathcal{S} = \{tat^{-1}w_a^{-1} : a \in \mathcal{A}\}$ for some words $w_a \in \mathcal{A}^{\pm *}$. For each $k \in \mathbb{Z}$, let $\Phi_k : \mathcal{A}^{\pm *} \to \mathcal{A}^{\pm *}$ be an endomorphism that lifts $\theta^k : K \to K$ and

commutes with the inversion involution of $\mathcal{A}^{\pm *}$. We take Φ_0 to be the identity. Define the following collections of words in $\mathcal{A}^{\pm *}$:

$$\overline{\mathcal{R}} = \{\Phi_k(r) \,:\, r \in \mathcal{R}, k \in \mathbb{Z}\}$$
$$\overline{\mathcal{S}} = \{\Phi_{k+1}(a)\Phi_k(w_a)^{-1} \,:\, a \in \mathcal{A}, k \in \mathbb{Z}\}.$$

Note that each word in $\overline{\mathcal{R}} \cup \overline{\mathcal{S}}$ is null-homotopic in K. Define $\mathcal{P}_K^{\infty} = \langle \mathcal{A} \,|\, \overline{\mathcal{R}}, \overline{\mathcal{S}} \rangle$ and define an index $\|\cdot\|$ on $\overline{\mathcal{R}} \cup \overline{\mathcal{S}}$ by setting $\|\omega\|$ to be the minimal value of $|k|$ such that either $\omega \equiv \Phi_k(r)$ for some $r \in \mathcal{R}$ or $\omega \equiv \Phi_{k+1}(a)\Phi_k(w_a)^{-1}$ for some $a \in \mathcal{A}$.

Theorem 7.4. *K is presented by $\mathcal{P}_K^{\infty}$. Furthermore, if (α, ρ) is an area-radius pair for $\mathcal{P}_\Gamma$ then it is also an area-penetration pair for the indexed presentation $(\mathcal{P}_K^{\infty}, \|\cdot\|)$.*

The utility of Theorem 7.4 is that if one can demonstrate that each word in $\overline{\mathcal{R}} \cup \overline{\mathcal{S}}$ is null-homotopic over some finite presentation $\mathcal{P}_K$, then it will follow that $\mathcal{P}_K$ presents K. Furthermore, by applying Proposition 6.2 one can obtain an upper bound on the Dehn function of $\mathcal{P}_K$.

The following slightly weaker version of Theorem 7.4 will actually be sufficient for our purposes. This result also has the advantage that its proof can be seen intuitively in the language of pictures. However, we wish to avoid having to prove the equivalence given in Propositions 3.17 and 6.4 between algebraically and pictorially defined area-radius and area-penetration pairs. We thus give a proof of Theorem 7.5 in the language of pictures and follow this with an algebraic proof of Theorem 7.4.

Theorem 7.5. *K is presented by $\mathcal{P}_K^{\infty}$. Furthermore, if (α, ρ) is an area-radius pair for $\mathcal{P}_\Gamma$ then there exist functions $\alpha', \rho' : \mathbb{N} \to \mathbb{N}$ with $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$ such that (α', ρ') is an area-penetration pair for the indexed presentation $(\mathcal{P}_K^{\infty}, \|\cdot\|)$.*

Proof. Let $w \in \mathcal{A}^{\pm *}$ be a null-homotopic word of length at most n. By Proposition 3.17 there exists a pictorial area-radius pair (α', ρ') for $\mathcal{P}_\Gamma$ such that $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$. Let $\mathbb{P}$ be a $\mathcal{P}_\Gamma$-picture with boundary word w such that $\operatorname{Area} \mathbb{P} \leq \alpha'(n)$ and $\operatorname{Rad} \mathbb{P} \leq \rho'(n)$. Say $\mathbb{P}$ has ambient disc D, basepoint b, relator discs $D_1, \ldots, D_m$ and arcs $\gamma_1, \ldots, \gamma_l$.

We now describe how to assign to each complementary region C of $\mathbb{P}$ an element $g(C)$ of Γ. If σ is a transverse path between points in $\operatorname{Back} \mathbb{P}$ then reading along σ defines a word $W(\sigma) \in (\mathcal{A} \cup \{t\})^{\pm *}$, where we understand that if σ crosses an arc labelled x in the direction of its normal orientation then we read x, and if σ crosses the arc in the opposite direction to its normal orientation then we read x^{-1}. By [36, Theorem 2.3] if $p_1, p_2 \in \operatorname{Back} \mathbb{P}$ and τ and τ' are transverse paths from p_1 to p_2 then $W(\tau)$ and $W(\tau')$ represent the same element in Γ. Given a point $p \in \operatorname{Back} \mathbb{P}$ define $g(p)$ to be the element of Γ represented by a transverse path σ from b to p. If p' lies in the same complementary region C as p then we can adjoin to σ a path from p to p' lying wholly in C to obtain a transverse path σ' from b to p' with $W(\sigma) = W(\sigma')$. Thus $g(p) = g(p')$ and we can define $g(C)$ to be this element of Γ.

By an $\mathcal{A}$-arc of $\mathbb{P}$ we mean an arc labelled by a letter in $\mathcal{A}$. We now show how to assign a height $h(\gamma) \in \mathbb{Z}$ to each $\mathcal{A}$-arc γ. Let $\overline{t}$ be the image of t under the quotient homomorphism $q : \Gamma \to \Gamma/K \cong \mathbb{Z}$ and define the height $h(C)$ of a

complementary region C of $\mathbb{P}$ to be the exponent of $\bar{t}$ in $q(g(C))$. Now suppose that γ is an arc of $\mathbb{P}$ labelled by the letter $a \in \mathcal{A}$. Say γ lies in the boundary of the complementary regions C_1 and C_2, which may or may not be distinct. We will show that $h(C_1) = h(C_2)$ and define $h(\gamma)$ to be this number. Let σ_1 be a transverse path from b to a point $p_1 \in C_1$ and let τ be a transverse path from p_1 to a point $p_2 \in C_2$ which intersects γ exactly once and intersects no other arcs of $\mathbb{P}$. Then the composition σ_2 of σ_1 and τ is a transverse path from b to p_2 with $W(\sigma_2) = W(\sigma_1)W(\tau) = W(\sigma_1)a^{\pm 1}$ in $(\mathcal{A} \cup \{t\})^{\pm *}$. Thus $g(C_2) = g(C_2)a^{\pm 1}$ in Γ and so $h(C_2) = h(C_1)$.

Note that for each complementary region C we can choose a transverse path from b to a point in C with intersection number at most $\operatorname{Rad}\mathbb{P}$ and so $|h(C)| \leq \operatorname{Rad}\mathbb{P}$. It follows that for all $\mathcal{A}$-arcs γ one similarly has $|h(\gamma)| \leq \operatorname{Rad}\mathbb{P}$.

We now modify $\mathbb{P}$ to produce a $\mathcal{P}_K^{\infty}$-picture $\overline{\mathbb{P}}$ for the word w. This is done by deleting each $\mathcal{A}$-arc γ_i labelled by a letter a and replacing it by a collection of $l_i := \left|\Phi_{h(\alpha_i)}(a)\right|$ parallel arcs $\gamma_i^1, \ldots, \gamma_i^{l_i}$ labelled by the letters of the word $\Phi_{h(\gamma_i)}(a)$. We now describe precisely what we mean by this. Say γ_i joins $\partial\Lambda_i^{\iota}$ to $\partial\Lambda_i^{\tau}$, where $\Lambda_i^{\iota}, \Lambda_i^{\tau} \in \{D, D_1, \ldots, D_m\}$. Let N_i^{ι} and N_i^{τ} be neighbourhoods of $\gamma_i \cap \Lambda_i^{\iota}$ and $\gamma_i \cap \Lambda_i^{\tau}$ in $\partial\Lambda_i^{\iota}$ and $\partial\Lambda_i^{\tau}$ respectively. We choose N_i^{ι} and N_i^{τ} to be homoeomorphic to the unit interval and to be disjoint from all basepoints and all other arcs of $\mathbb{P}$. Each γ_i^j joins N_i^{ι} to N_i^{τ} and we choose them so as they are all disjoint and their interiors are disjoint from $\cup_{k=1}^m D_k$. We orientate and label the arcs γ_i^j so as reading along N_i^{ι} in the direction of the orientation of γ_i gives the word $\Phi_{h(\gamma_i)}(a)$. The picture $\overline{\mathbb{P}}$ is now completed by deleting all the arcs γ_i labelled by the letter t.

If a disc D_i had label $r \in \mathcal{R}^{\pm 1}$ in $\mathbb{P}$ then all the arcs incident with D_i in $\mathbb{P}$ had the same height h. Thus the corresponding disc in $\overline{\mathbb{P}}$ has label $\Phi_h(r) \in \overline{R}^{\pm 1}$ for some h with $|h| \leq \operatorname{Rad}\mathbb{P}$. If the disc D_i had the label $\left(tat^{-1}w_a^{-1}\right)^{\pm 1} \in \mathcal{S}^{\pm 1}$ in $\mathbb{P}$ then the incident arc labelled a had height h and the incident arcs labelled by the letters of w_a had height $h-1$, for some $h \in \mathbb{Z}$. Thus the corresponding disc in $\overline{\mathbb{P}}$ has label $\left(\Phi_h(a)\Phi_{h-1}(w_a)^{-1}\right)^{\pm 1} \in \overline{\mathcal{S}}^{\pm 1}$ for some h with $|h|$ and $|h-1|$ at most $\operatorname{Rad}\mathbb{P}$.

By a boundary arc of $\mathbb{P}$ we will mean an arc with at least one of its endpoints lying in ∂D. Note that all boundary arcs of $\mathbb{P}$ are $\mathcal{A}$-arcs. If C is a complementary region of $\mathbb{P}$ with the boundary of its closure intersecting ∂D non-trivially, then there exists a transverse path in $\mathbb{P}$ from b to C which intersects only boundary arcs. Thus C has zero height. It follows that all the boundary arcs of $\mathbb{P}$ have zero height and hence that the boundary label of $\overline{\mathbb{P}}$ is $\Phi_0(w) \equiv w$. Thus $\overline{\mathbb{P}}$ is a $\mathcal{P}_K^{\infty}$-picture for the word w, with $\operatorname{Area}\overline{\mathbb{P}} = \operatorname{Area}\mathbb{P}$ and with each relator $z \in (\overline{\mathcal{R}} \cup \overline{\mathcal{S}})^{\pm 1}$ labelling a disc of $\overline{\mathbb{P}}$ having $\|z\| \leq \operatorname{Rad}\mathbb{P}$.

Since the word w was arbitrary it follows that $\mathcal{P}_K^{\infty}$ presents K and has (α', ρ') as a pictorial area-penetration pair. By Proposition 6.4 it follows that (α', ρ') is also an area-penetration pair for $\mathcal{P}_K^{\infty}$. $\square$

Proof of Theorem 7.4. Let $w \in \mathcal{A}^{\pm *}$ be a null-homotopic word of length at most n and let $(x_i, z_i)_{i=1}^m$ be a $\mathcal{P}_\Gamma$-expression for w with $m \leq \alpha(n)$ and with $|x_i| \leq \rho(n)$ for each i.

We write $h(u)$ for the exponent sum in the letter t of a word $u \in (\mathcal{A} \cup \{t\})^{\pm *}$ and define $\widetilde{N}$ to be the submonoid of $(\mathcal{A} \cup \{t\})^{\pm *}$ consisting of all those words u with $h(u) = 0$. Define $\mathcal{X}$ to be the set of words $\{t^k a t^{-k} : a \in \mathcal{A}, k \in \mathbb{Z}\} \leq$

$(\mathcal{A} \cup \{t\})^{\pm *}$. Let L be the submonoid of $\widetilde{N}$ generated by $\mathcal{X}^{\pm 1}$ and note that L is free on this basis. If $u \in \widetilde{N}$ write $\Lambda(u)$ for the unique word in L which is freely equal to u in $F(\mathcal{A} \cup \{t\})$ and freely reduced as an element of $F(\mathcal{X})$. For each $i \in \{1, \ldots, m\}$, define $\overline{x}_i \equiv \Lambda(x_i t^{-h(x_i)})$ and $\overline{z}_i = \Lambda(t^{h(x_i)} z_i t^{-h(x_i)})$. Define $\sigma \equiv \prod_{i=1}^m \overline{x}_i \overline{z}_i \overline{x}_i^{-1}$ and note that $w \overset{\text{fr}}{=} \sigma$ in $F(\mathcal{A} \cup \{t\})$.

Define a homomorphism $\Psi : L \to \mathcal{A}^{\pm *}$, which commutes with the inversion involution of L, by mapping $t^k a t^{-k} \mapsto \Phi_k(a)$. Let N be the kernel of the homomorphism $F(\mathcal{A} \cup \{t\}) \to \mathbb{Z}$ defined by mapping t to 1 and each $a \in \mathcal{A}$ to 0, and note that N is free with basis the image of $\mathcal{X}$. Thus Ψ descends to a homomorphism $N \to F(\mathcal{A})$ and since $w \overset{\text{fr}}{=} \sigma$ in N we have that $\Psi(w) \overset{\text{fr}}{=} \Psi(\sigma)$ in $F(\mathcal{A})$. Observe that $\Psi(\sigma) \equiv \prod_{i=1}^m \Psi(\overline{x}_i)\Psi(\overline{z}_i)\Psi(\overline{x}_i)^{-1}$ and $\Psi(w) \equiv w$ since w contains no occurrence of the letter t.

If $z_i \equiv a_1 \ldots a_l \in \mathcal{R}$ then $\overline{z}_i \equiv t^k a_1 t^{-k} \ldots t^k a_l t^{-k}$ for some $k \in \mathbb{Z}$ with $|k| = |h(x_i)| \leq |x_i|$. Thus $\Psi(\overline{z}_i) \equiv \Phi_k(z_i)$ where $|k| \leq \rho(n)$. If $z_i \equiv tat^{-1}a_1 \ldots a_l \in \mathcal{S}$ then $\overline{z}_i \equiv t^{k+1} a t^{-k-1} t^k a_1 t^{-k} \ldots t^k a_l t^{-k}$ for some $k \in \mathbb{Z}$ with $|k| = |h(x_i)| \leq |x_i|$. Thus $\Psi(\overline{z}_i) \equiv \Phi_{k+1}(a)\Phi_k(w_a)^{-1}$ where $\min\{|k+1|, |k|\} \leq |k| \leq \rho(n)$. In either case we have that $\Psi(\overline{z}_i) \in \overline{\mathcal{R}} \cup \overline{\mathcal{S}}$ and $\|\Psi(\overline{z}_i)\| \leq \rho(n)$. Thus $(\Psi(\overline{x}_i), \Psi(\overline{z}_i))_{i=1}^m$ is a $\mathcal{P}_K^\infty$-expression for w and, since w was arbitrary, we see that $\mathcal{P}_K^\infty$ presents K and that (α, ρ) is an area-penetration pair for $\mathcal{P}_K^\infty$. $\square$

8 Amalgamated products

In this section we present a method for giving lower bounds on the Dehn functions of amalgamated products. Specifically we will be concerned with finitely presented amalgamated products $\Gamma = G_1 *_H G_2$ of finitely generated groups G_1 and G_2 over a finitely generated subgroup H which is proper in each G_i.

Suppose each G_i is presented by $\langle \mathcal{A}_i \,|\, \mathcal{R}_i \rangle$, with $\mathcal{A}_i$ finite. Note that we are at liberty to choose the $\mathcal{A}_i$ so as each $a \in \mathcal{A}_i$ represents an element of $G_i \smallsetminus H$. Indeed, since H is proper in G_i, there exists some $a' \in \mathcal{A}_i$ representing an element of $G_i \smallsetminus H$ and we can replace each other element $a \in \mathcal{A}_i$ by $a'a$ if necessary. Let $\mathcal{B}$ be a finite generating set for H and for each $b \in \mathcal{B}$ choose words $u_b \in \mathcal{A}_1^{\pm *}$ and $v_b \in \mathcal{A}_2^{\pm *}$ which equal b in Γ. Define $\mathcal{E} \subseteq (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B})^{\pm *}$ to be the finite collection of words $\{bu_b^{-1}, bv_b^{-1} \;:\; b \in \mathcal{B}\}$. Then, since Γ is finitely presented, there exist finite subsets $\mathcal{R}_1' \subseteq \mathcal{R}_1$ and $\mathcal{R}_2' \subseteq \mathcal{R}_2$ such that Γ is finitely presented by

$$\mathcal{P} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{B} \,|\, \mathcal{R}_1', \mathcal{R}_2', \mathcal{E} \rangle.$$

Theorem 8.1. *Let $w \in \mathcal{A}_1^{\pm *}$ be a word representing an element $h \in H$ and let $u \in \mathcal{A}_1^{\pm *}$ and $v \in \mathcal{A}_2^{\pm *}$ be words representing elements $\alpha \in G_1 \smallsetminus H$ and $\beta \in G_2 \smallsetminus H$ respectively. If $[\alpha, h] = [\beta, h] = 1$ then*

$$\mathrm{Area}_{\mathcal{P}}([w, (uv)^n]) \geq 2n \, \mathrm{d}_{\mathcal{B}}(1, h)$$

where $\mathrm{d}_{\mathcal{B}}$ is the word metric on H associated to the generating set $\mathcal{B}$.

Proof. Let Δ be a $\mathcal{P}$-van Kampen diagram for the null-homotopic word $[w, (uv)^n]$. For each $i = 1, 2, \ldots, n$ define p_i to be the vertex in $\partial\Delta$ such that the anticlockwise path in $\partial\Delta$ from the basepoint around to p_i is labelled by the word $w(uv)^{i-1}u$. Similarly define q_i to be the vertex in $\partial\Delta$ such that the

anticlockwise path in $\partial\Delta$ from the basepoint around to q_i is labelled by the word $w(uv)^n w^{-1}(uv)^{i-n}v^{-1}$. We will show that for each i there is a $\mathcal{B}$-path (i.e. an edge path in Δ labelled by a word in the letters $\mathcal{B}$) from p_i to q_i.

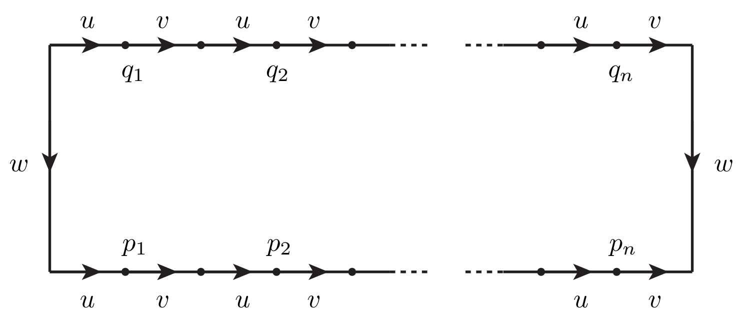

Figure 1: The van Kampen diagram Δ

We assume that the reader is familiar with Bass-Serre theory, as exposited in [40]. Let T be the Bass-Serre tree associated to the splitting $G_1 *_H G_2$. This consists of an edge gH for each coset Γ/H and a vertex gG_i for each coset Γ/G_i. The edge gH has initial vertex gG_1 and terminal vertex gG_2. We will construct a continuous (but non-combinatorial) map $\Delta \to T$ as the composition of the natural map $\Delta \to Cay^2(\mathcal{P})$ with the map $f : Cay^2(\mathcal{P}) \to T$ defined below.

There is a natural left action of Γ on each of $Cay^2(\mathcal{P})$ and T and we construct f to be equivariant with respect to this as follows. Let m be the midpoint of the edge H of T and define f to map the vertex $g \in Cay^2(\mathcal{P})$ to the point $g \cdot m$, the midpoint of the edge gH. Define f to map the edge of $Cay^2(\mathcal{P})$ labelled $a \in \mathcal{A}_i$ joining vertices g and ga to the geodesic segment joining $g \cdot m$ to $ga \cdot m$. Since $a \notin H$ this segment is an embedded arc of length 1 whose midpoint is the vertex gG_i. Define f to collapse the edge in $Cay^2(\mathcal{P})$ labelled $b \in \mathcal{B}$ joining vertices g and gb to the point $g \cdot m = gb \cdot m$. This is well defined since $gH = gbH$. This completes the definition of f on the 1-skeleton of Δ; we now extend f over the 2-skeleton.

Let c be a 2-cell in $Cay^2(\mathcal{P})$ and let g be some vertex in its boundary. Assume that c is metrised so as to be convex and let l be some point in its interior. The form of the relations in $\mathcal{P}$ ensures that the boundary label of c is a word in the letters $\mathcal{A}_i \cup \mathcal{B}$ for some i and so every vertex in ∂c is labelled gg' for some $g' \in G_i$. Thus f as so far defined maps ∂c into the ball of radius $1/2$ centred on the vertex gG_i; we extend f to the interior of c by defining it to map the geodesic segment $[l, p]$, where $p \in \partial c$, to the geodesic segment $[gG_i, f(p)]$. This is independent of the vertex $g \in \partial c$ chosen and makes f continuous since geodesics in a tree vary continuously with their endpoints. We now define $\bar{f} : \Delta \to T$ to be the map given by composing f with the label-preserving map $\Delta \to Cay^2(\mathcal{P})$ which sends the basepoint of Δ to the vertex $1 \in Cay^2(\mathcal{P})$.

Since w commutes with u and v we have that $\bar{f}(p_i) = w(uv)^{i-1}u \cdot m = (uv)^{i-1}u \cdot m = \bar{f}(q_i)$; define S to be the preimage under $\bar{f}$ of this point. By construction, the image of the interior of each 2-cell in Δ and the image of the interior of each $\mathcal{A}_i$-edge is disjoint from $\bar{f}(p_i)$. Thus S consists of vertices and $\mathcal{B}$-edges and so finding a $\mathcal{B}$-path from p_i to q_i reduces to finding a path in S connecting these vertices. Let s_i and t_i be the vertices of $\partial\Delta$ immediately

preceding and succeeding p_i in the boundary cycle. Unless $h = 1$, in which case the theorem is trivial, the form of the word $[w, (uv)^n]$, together with the normal form theorem for amalgamated products, implies that all the vertices p_i, s_i and t_i lie in the boundary of the same disc component D of Δ. Furthermore since u and v are words in the letters $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively the points $f(s_i)$ and $f(t_i)$ are separated in T by $f(p_i)$. Thus s_i and t_i are separated in D by S and so there exists an edge path γ_i in S from p_i to some other vertex $r_i \in \partial D$. Since γ_i is a $\mathcal{B}$-path it follows that the word labelling the sub-arc of the boundary cycle of Δ from p_i to r_i represents an element of H, and, by considering subwords of $[w, (uv)^n]$, we see that the only possibility is that $r_i = q_i$. Thus for each $i = 1, \ldots, n$ the path γ_i gives the required $\mathcal{B}$-path connecting p_i to q_i. We choose each γ_i to contain no repeated edges.

For $i \neq j$ the two paths γ_i and γ_j are disjoint since if they intersected there would be a $\mathcal{B}$-path joining p_i to p_j and thus the word labelling the subarc of the boundary cycle from p_i to p_j would represent an element of H. Observe that no two edges in any of the paths $\gamma_1, \ldots, \gamma_n$ lie in the boundary of the same 2-cell in Δ since each relation in $\mathcal{P}$ contains at most one occurrence of a letter in $\mathcal{B}$. Because the word labelling $\partial\Delta$ contains no occurrences of a letter in $\mathcal{B}$ the interior of each edge of a path γ_i lies in the interior of Δ and thus in the boundary of two distinct 2-cells. Since each path γ_i contains no repeated edges we therefore obtain the bound $\mathrm{Area}(\Delta) \geq \sum_{i=1}^{n} 2|\gamma_i|$. But the word labelling each γ_i is equal to h in Γ and so the length of γ_i is at least $\mathrm{d}_{\mathcal{B}}(1, h)$ whence we obtain the required inequality. $\square$

9 Fibre products

Definition 9.1. Given a homomorphism $p : \Gamma \to Q$, the *(untwisted) fibre product* of p is defined to be the subgroup $\{(\gamma_1, \gamma_2) : p(\gamma_1) = p(\gamma_2)\} \leq \Gamma \times \Gamma$.

Recall the following result of Baumslag, Bridson, Miller and Short.

Theorem 9.2 (The 1-2-3 Theorem [4])**.** *Let $1 \to N \to \Gamma \xrightarrow{p} Q \to 1$ be a short exact sequence of groups. Suppose that N is finitely generated, Γ is finitely presented and Q is of type F_3. Then the fibre product of p is finitely presented.*

Definition 9.3. Given a pair of homomorphisms $p_i : \Gamma_i \to Q$, $i = 1, 2$, the *(twisted) fibre product* of p_1 and p_2 is defined to be the subgroup $\{(\gamma_1, \gamma_2) : p_1(\gamma_1) = p_2(\gamma_2)\} \leq \Gamma_1 \times \Gamma_2$.

In this section we prove a generalisation of the 1-2-3 theorem which covers twisted fibre products.

Theorem 9.4. *For each $i = 1, 2$, let $1 \to N_i \to \Gamma_i \xrightarrow{p_i} Q \to 1$ be a short exact sequence of groups. Suppose that N_1 is finitely generated, Γ_1 and Γ_2 are finitely presented, and Q is of type F_3. Then the fibre product of p_1 and p_2 is finitely presented.*

Note that we do not need to make any assumptions about N_2. The proof of Theorem 9.4 given below closely follows the proof of Theorem 9.2 given in [4]. We will require the following lemma.

Lemma 9.5. *For each $i = 1, 2$, let $p_i : \Gamma_i \to Q$ be a surjective homomorphism. Suppose that Γ_1 and Γ_2 are finitely generated and that Q is finitely presented. Then the fibre product P of p_1 and p_2 is finitely generated. If α is an isoperimetric function for some finite presentation of Q then the distortion function Δ of P in $\Gamma_1 \times \Gamma_2$ satisfies $\Delta \preceq \alpha$.*

More specifically, let $\mathcal{X}_1$ be a finite generating set for Γ_1 and let $\mathcal{X}$ be the image of $\mathcal{X}_1$ in Q. Let $\mathcal{X}_2$ be a choice of lifts of the elements of $\mathcal{X}$ under p_2 and let $\mathcal{A} \subseteq \ker p_2$ be a finite collection of elements such that Γ_2 is generated by $\mathcal{A} \cup \mathcal{X}_2$. Let $\langle \mathcal{X} \mid \mathcal{R} \rangle$ be a finite presentation for Q. Then P is generated by the union of the following sets of elements:

$$\begin{aligned}
\overline{\mathcal{X}} &= \{(x_1, x_2) : x_i \in \mathcal{X}_i, p_1(x_1) = p_2(x_2)\};\\
\overline{\mathcal{A}} &= \{(1, a) : a \in \mathcal{A}\};\\
\overline{\mathcal{R}} &= \{(r(\mathcal{X}_1), 1) : r(\mathcal{X}) \in \mathcal{R}\}.
\end{aligned}$$

Remark 9.6. Note that the bound on Δ is only defined up to $\simeq$-equivalence, not the stronger $\approx$-equivalence usually used with distortion functions.

Proof of Lemma 9.5. Fix compatible orderings on $\mathcal{X}$, $\mathcal{X}_1$, $\mathcal{X}_2$ and $\overline{\mathcal{X}}$; and on $\mathcal{A}$ and $\overline{\mathcal{A}}$. By Lemma 3.11, the Dehn function δ of $\langle \mathcal{X} \mid \mathcal{R} \rangle$ satisfies $\delta \preceq \alpha$.

Let $w = w(\mathcal{X}_1, \mathcal{X}_2, \mathcal{A})$ be a word representing an element γ of P. Then $w \overset{P}{=} w(\mathcal{X}_1, \emptyset, \emptyset)w(\emptyset, \mathcal{X}_2, \mathcal{A}) \overset{\text{fr}}{=} w(\mathcal{X}_1, \emptyset, \emptyset)w^{-1}(\emptyset, \mathcal{X}_1, \emptyset)w(\emptyset, \mathcal{X}_1, \emptyset)w(\emptyset, \mathcal{X}_2, \mathcal{A}) \overset{P}{=} w(\mathcal{X}_1, \emptyset, \emptyset)w^{-1}(\emptyset, \mathcal{X}_1, \emptyset)w(\emptyset, \overline{\mathcal{X}}, \overline{\mathcal{A}})$. Define $w_1(\mathcal{X}_1) \equiv w(\mathcal{X}_1, \emptyset, \emptyset)w^{-1}(\emptyset, \mathcal{X}_1, \emptyset)$ and note that $|w_1| \leq |w|$. Furthermore $p_1(w_1(\mathcal{X}_1))$ is trivial in Q and so $w_1(\mathcal{X})$ is null-homotopic. Define $L = \max\{|r| : r \in \mathcal{R}\}$. By Lemma 5.1, there exist words $r_1, \ldots r_n \in \mathcal{R}^{\pm 1}$ and words $x_0, \ldots, x_n \in \mathcal{X}^{\pm *}$ with $n \leq \delta(|w_1|)$ and $\sum |x_i| \leq |w_1| + 2Ln$ so that $w_1(\mathcal{X}) \overset{\text{fr}}{=} x_0 r_1 x_1 \ldots r_n x_n$ and the word $x_0 \ldots x_n \overset{\text{fr}}{=} \emptyset$. Thus $w_1(\mathcal{X}_1) \overset{\text{fr}}{=} x_0(\mathcal{X}_1)r_1(\mathcal{X}_1) \ldots r_n(\mathcal{X}_1)x_n(\mathcal{X}_1) \overset{P}{=} x_0(\overline{\mathcal{X}})(r_1(\mathcal{X}_1), 1) \ldots (r_n(\mathcal{X}_1), 1)x_n(\overline{\mathcal{X}})$ and so γ is represented by a word in the letters $\overline{\mathcal{X}}$, $\overline{\mathcal{A}}$ and $\overline{\mathcal{R}}$ of length at most $(2L+1)\delta(|w|) + 2|w|$. Thus $\Delta \preceq \delta \preceq \alpha$. $\square$

Proof of Theorem 9.4. Let $\mathcal{X}_1$ be a finite ordered generating set for Γ_1 and let $\mathcal{X}$ be the image of $\mathcal{X}_1$ in Q. Then there is an induced ordering on $\mathcal{X}$ and $\mathcal{X}$ generates Q. Let $\mathcal{A}_1$ be a finite ordered generating set for N_1. For each $a \in \mathcal{A}_1$, $x \in \mathcal{X}_1$ and $\epsilon \in \{\pm 1\}$, choose a word $w_{ax\epsilon} \in \mathcal{A}_1^{\pm *}$ such that $x^\epsilon a x^{-\epsilon} = w_{ax\epsilon}$ in Γ_1. Let $\langle \mathcal{X} \mid \mathcal{R} \rangle$ be a finite presentation for Q and for each $r = r(\mathcal{X}) \in \mathcal{R}$ choose a word $w_r \in \mathcal{A}_1^{\pm *}$ such that $r(\mathcal{X}_1) = w_r$ in Γ_1. Define

$$\mathcal{R}_1 = \left\{ x^\epsilon a x^{-\epsilon} w_{ax\epsilon}^{-1} \,:\, a \in \mathcal{A}_1, x \in \mathcal{X}_1, \epsilon \in \{\pm 1\} \right\}$$

and

$$\mathcal{R}_2 = \left\{ r(\mathcal{X}_1) w_r^{-1} \,:\, r \in \mathcal{R} \right\}.$$

If $w = w(\mathcal{A}_1, \mathcal{X}_1)$ is null-homotopic in Γ_1 then, modulo relators in $\mathcal{R}_1$, w is equal to a word of the form $u(\mathcal{A}_1)v(\mathcal{X}_1)$. The word $v(\mathcal{X})$ is null-homotopic in Q and hence there is a free equality $v(\mathcal{X}_1) = \prod \rho_i(\mathcal{X}_1) r_i(\mathcal{X}_1) \rho_i(\mathcal{X}_1)^{-1}$ for some $r_i = r_i(\mathcal{X}) \in \mathcal{R}$ and some words ρ_i. Thus, modulo relators in $\mathcal{R}_1$ and $\mathcal{R}_2$, w is equal to a word in the letters $\mathcal{A}_1$. It follows that there exists a finite collection of relations $\mathcal{R}_3 \subset \mathcal{A}_1^{\pm *}$ such that Γ_1 is presented by $\langle \mathcal{A}_1, \mathcal{X}_1 \mid \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \rangle$.

Let the finite set $\mathcal{X}_2 \subset \Gamma_2$ be a choice of lifts of the elements of $\mathcal{X}$ under p_2 ordered compatibly with $\mathcal{X}$. Then there exists a finite ordered collection of

elements $\mathcal{A}_2 \subset N_2$ so that $\mathcal{X}_2 \cup \mathcal{A}_2$ generates Γ_2. Note that $\mathcal{A}_2$ may not generate N_2. Let $\langle \mathcal{A}_2, \mathcal{X}_2 \,|\, \mathcal{R}_4 \rangle$ be a finite presentation for Γ_2.

By the argument in the proof of Lemma 9.5, the fibre product P of p_1 and p_2 is generated by the union of the following sets of elements:

$$\begin{aligned} \overline{\mathcal{X}} &= \{(x_1, x_2) \,:\, x_i \in \mathcal{X}_i, p_1(x_1) = p_2(x_2)\}; \\ \overline{\mathcal{A}}_1 &= \{(a, 1) \,:\, a \in \mathcal{A}_1\}; \\ \overline{\mathcal{A}}_2 &= \{(1, a) \,:\, a \in \mathcal{A}_2\}. \end{aligned}$$

Order the elements of $\overline{\mathcal{X}}$, $\overline{\mathcal{A}}_1$ and $\overline{\mathcal{A}}_2$ compatibly with the $\mathcal{X}_i$, $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. We now define some relations which hold amongst these generators:

$$\begin{aligned} \mathcal{S}_1 &= \big\{[\bar{a}_1, \bar{a}_2] \,:\, \bar{a}_i \in \overline{\mathcal{A}}_i\big\} \\ \mathcal{S}_2 &= \big\{r(\overline{\mathcal{X}}, \overline{\mathcal{A}}_1) \,:\, r = r(\mathcal{X}_1, \mathcal{A}_1) \in \mathcal{R}_1\big\} \\ \mathcal{S}_3 &= \big\{r(\overline{\mathcal{A}}_1) \,:\, r = r(\mathcal{A}_1) \in \mathcal{R}_3\big\} \\ \mathcal{S}_4 &= \big\{[r(\overline{\mathcal{X}}, \overline{\mathcal{A}}_1), \bar{a}] \,:\, r = r(\mathcal{X}_1, \mathcal{A}_1) \in \mathcal{R}_2, \bar{a} \in \overline{\mathcal{A}}_1\big\} \end{aligned}$$

For each $r = r(\mathcal{X}_2, \mathcal{A}_2) \in \mathcal{R}_4$, choose a word $w_r \in \overline{\mathcal{A}}_1^{\pm *}$ so that $r(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2) = w_r$ in $\Gamma_1 \times \Gamma_2$. Then we can define the set of relations

$$\mathcal{S}_5 = \big\{r(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2) w_r(\overline{\mathcal{A}}_1)^{-1} \,:\, r(\mathcal{X}_2, \mathcal{A}_2) \in \mathcal{R}_4\big\}.$$

Let Σ be a finite generating set of Peiffer sequences for $\pi_2(Q)$ as a Q-module. Each $\sigma \in \Sigma$ is a sequence $(u_1 r_1 u_1^{-1}, \ldots, u_n r_n u_n^{-1})$ where each $r_i = r_i(\mathcal{X}) \in \mathcal{R}$, each $u_i = u_i(\mathcal{X})$ is a word in $\mathcal{X}^{\pm *}$ and the word

$$\zeta_\sigma(\mathcal{X}) = \prod_i u_i(\mathcal{X}) r_i(\mathcal{X}) u_i(\mathcal{X})^{-1}$$

is freely equal to the empty word. Observe that, modulo relations in $\mathcal{R}_2$, the word $\zeta_\sigma(\mathcal{X}_1)$ is equal to

$$\prod_i u_i(\mathcal{X}_1) w_{r_i}(\mathcal{A}_1) u_i(\mathcal{X}_1)^{-1}$$

and this is equal, modulo relations in $\mathcal{R}_1$, to a word $Z_\sigma = Z_\sigma(\mathcal{A}_1)$. We define

$$\mathcal{S}_6 = \big\{Z_\sigma(\overline{\mathcal{A}}_1) \,:\, \sigma \in \Sigma\big\}.$$

We claim that P is presented by $\langle \overline{\mathcal{X}}, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2 \,|\, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6 \rangle$. Indeed suppose that $w = w(\overline{\mathcal{X}}, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2)$ is null-homotopic. Then the relations in $\mathcal{S}_1$ and $\mathcal{S}_2$ are sufficient to convert w to a word $w_1(\overline{\mathcal{A}}_1) w_2(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2)$. Projecting onto the factor Γ_2 demonstrates that the word $w_2(\mathcal{X}_2, \mathcal{A}_2)$ is null-homotopic. There thus exists a free equality

$$w_2(\mathcal{X}_2, \mathcal{A}_2) \stackrel{\text{fr}}{=} \prod_i u_i(\mathcal{X}_2, \mathcal{A}_2) r_i(\mathcal{X}_2, \mathcal{A}_2) u_i(\mathcal{X}_2, \mathcal{A}_2)^{-1}$$

for some words u_i and some relations $r_i \in \mathcal{R}_4$ and hence a free equality

$$w_2(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2) \stackrel{\text{fr}}{=} \prod_i u_i(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2) r_i(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2) u_i(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2)^{-1}.$$

The relations in $\mathcal{S}_5$ and in $\mathcal{S}_1$ and $\mathcal{S}_2$ are sufficient to convert $w_2(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2)$ to the word

$$\prod_i u_i(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2) w_{r_i}(\overline{\mathcal{A}}_1)^{-1} u_i(\overline{\mathcal{X}}, \overline{\mathcal{A}}_2)^{-1}$$

and thence to some word in the letters $\overline{\mathcal{A}}_1$. The word $w(\overline{\mathcal{X}}, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2)$ can thus be converted to a word $w' = w'(\overline{\mathcal{A}}_1)$. We now recall the following result of Baumslag, Bridson, Miller and Short:

Lemma 9.7 ([4]). *A word $v = v(\mathcal{A}_1)$ is null-homotopic in Γ_1 if and only if it is freely equal in $F(\mathcal{A}_1 \cup \mathcal{X}_1)$ to a product of conjugates of the following relations:*

- $\mathcal{R}_1$
- $\mathcal{R}_3$
- $\{Z_\sigma(\mathcal{A}_1) \,:\, \sigma \in \Sigma\}$
- $\{[r(\mathcal{A}_1, \mathcal{X}_1), a] \,:\, r \in \mathcal{R}_2, a \in \mathcal{A}_1\}$

Projecting $\Gamma_1 \times \Gamma_2$ onto the first factor demonstrates that $w'(\mathcal{A}_1)$ is null-homotopic in Γ_1 and hence there is an equality

$$w'(\mathcal{A}_1) \stackrel{\text{fr}}{=} \prod_i u_i(\mathcal{A}_1, \mathcal{X}_1) s_i(\mathcal{A}_1, \mathcal{X}_1) u_i(\mathcal{A}_1, \mathcal{X}_1)^{-1}$$

for some words u_i and some relations s_i from the sets given in Lemma 9.7. It follows that there is an equality

$$w'(\overline{\mathcal{A}}_1) \stackrel{\text{fr}}{=} \prod_i u_i(\overline{\mathcal{A}}_1, \overline{\mathcal{X}}) s_i(\overline{\mathcal{A}}_1, \overline{\mathcal{X}}) u_i(\overline{\mathcal{A}}_1, \overline{\mathcal{X}})^{-1}$$

where the $s_i = s_i(\overline{\mathcal{A}}_1, \overline{\mathcal{X}})$ are relations in $\mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_6$. This completes the proof of the claim. $\square$

10 Close fillings

Let H be a subgroup of a group G. In this section we establish criteria for H to be finitely generated or to be finitely presented. The utility of these criteria is that they are phrased entirely in terms of properties of generating sets (respectively presentations) for G, and so one avoids having to explicitly determine a generating set (respectively a presentation) for H. In the language of course geometry, the criteria amount to showing that H is coarsely connected (respectively coarsely simply connected) in G.

Suppose that G is finitely generated, and consider the vertices in the Cayley graph of G that represent elements of H. We will show that H is finitely generated if this set is coarsely connected. More explicitly, the criterion amounts to showing that every element of H can be represented by a word in the generators of G that, considered as a path in the Cayley graph of G, lies uniformly close to H. By considering the length of such words, one obtains a bound on the distortion of H in G.

If G is finitely presented then an analogous criterion will establish that H is itself finitely presented: this amounts to showing that an embedding of the

Cayley graph of H in the Cayley complex of G is coarsely simply connected. In the language of van Kampen diagrams one demonstrates that every null-homotopic edge loop in the Cayley 2-complex of G which lies close to H can be filled by a diagram which lies close to H. We translate this notion into the language of $\mathcal{P}$-expressions. By considering the areas of such expressions one obtains an isoperimetric function for H.

Definition 10.1. Let $\mathcal{X}$ be a generating set for G.

Given $g \in G$ define

$$d_{\mathcal{X}}(g, H) = \min_{h \in H} d_{\mathcal{X}}(g, h),$$

where $d_{\mathcal{X}}$ is the word metric on G associated to $\mathcal{X}$. Define the *departure* from H of a word $w \in \mathcal{X}^{\pm *}$ by

$$\mathrm{Dep}_{\mathcal{X}}(w, H) = \max_{0 \leq i \leq |w|} d_{\mathcal{X}}(w[i], H).$$

Proposition 10.2. *Let $\mathcal{X}$ be a finite generating set for the group G. Suppose that there exists a constant $K \geq 0$ such that for all $h \in H$ there exists a word $w_h \in \mathcal{X}^{\pm *}$ representing h in G with $\mathrm{Dep}_{\mathcal{X}}(w_h, H) \leq K$. Then there exists a finite generating set $\mathcal{Y}$ for H and the distortion function Δ of H in G with respect to $\mathcal{Y}$ and $\mathcal{X}$ satisfies*

$$\Delta(l) \leq \max\{|w_h| \,:\, d_{\mathcal{X}}(1, h) \leq l\}.$$

Proof. For each $g \in G$, choose an element $\gamma_g \in G$ such that $g\gamma_g^{-1} \in H$ and $d_{\mathcal{X}}(1, \gamma_g) = d_{\mathcal{X}}(g, H)$. Define a function $\Pi : G \times \mathcal{X}^{\pm 1} \to H$ by $\Pi(g, x) = \gamma_g x \gamma_{gx}^{-1}$. Define a function $\Psi : \mathcal{X}^{\pm *} \to H^{\pm *}$ by

$$\Psi(x_1 \dots x_n) = \Pi(1, x_1)\, \Pi(x_1, x_2)\, \Pi(x_1 x_2, x_3)\, \dots\, \Pi(x_1 \dots x_{n-1}, x_n)$$

and note that if $w \in \mathcal{X}^{\pm *}$ represents an element of H then $\Psi(w) = w$ in H.

Given $r \in \mathbb{N}$, define $N_r = \{g \in G \,:\, d_{\mathcal{X}}(g, H) \leq r\}$. Define $\mathcal{Y} = \Pi(N_K \times \mathcal{X}^{\pm 1}) \subseteq H$ and note that $\mathcal{Y}$ is finite since it is contained in the finite set $\{h \in H \,:\, d_{\mathcal{X}}(1, h) \leq 2K+2\}$. Observe that, for every $h \in H$, the word $\Psi(w_h) \in \mathcal{Y}^{\pm *}$ represents h and hence $\mathcal{Y}$ generates H. Furthermore $d_{\mathcal{Y}}(1, h) \leq |\Psi(w_h)| = |w_h|$ so Δ satisfies the given inequality. $\square$

Definition 10.3. Let $\mathcal{P}$ be a presentation of the group G. Define the *departure* from H of a $\mathcal{P}$-expression $\mathcal{E} = (x_i, r_i)_{i=1}^m$ to be

$$\mathrm{Dep}_{\mathcal{X}}(\mathcal{E}, H) = \max_{1 \leq i \leq m} \mathrm{Dep}_{\mathcal{X}}(x_i, H).$$

Proposition 10.4. *Let $\mathcal{P} = \langle \mathcal{X} \,|\, \mathcal{R} \rangle$ be a finite presentation of the group G and let $H \leq G$ be a finitely generated subgroup with finite generating set $\mathcal{Y}$.*

(1) Suppose that there exists a function $K : \mathbb{N} \to \mathbb{N}$ such that, for each null-homotopic word $w \in \mathcal{X}^{\pm *}$, there exists a $\mathcal{P}$-expression $\mathcal{E}_w$ for w with $\mathrm{Dep}_{\mathcal{X}}(\mathcal{E}_w, H) \leq K(\mathrm{Dep}_{\mathcal{X}}(w, H))$. Then there exists a finite set of words $\mathcal{S} \subseteq \mathcal{Y}^{\pm *}$ so that H is presented by $\mathcal{Q} = \langle \mathcal{Y} \,|\, \mathcal{S} \rangle$.

(2) *Suppose, in addition, that there exists a function* $\alpha : \mathbb{N} \to \mathbb{N}$ *so that* $\mathrm{Area}(\mathcal{E}_w) \le \alpha(|w|)$ *for each* w. *Then* α *is an isoperimetric function for* H.

(3) *Suppose, in addition, that there exists a function* $\rho : \mathbb{N} \to \mathbb{N}$ *so that* $\mathrm{Rad}(\mathcal{E}_w) \le \rho(|w|)$ *for each* w. *Then there exist functions* $\alpha', \rho' : \mathbb{N} \to \mathbb{N}$ *with* $\alpha' \simeq \alpha$ *and* $\rho' \simeq \rho$ *so that* (α', ρ') *is an area-radius pair for* $\mathcal{Q}$.

Proof. For each $y \in \mathcal{Y}$ choose a word $u_y \in \mathcal{X}^{\pm *}$ with $u_y = y$ in G. Define $L = \max\{\mathrm{Dep}_{\mathcal{X}}(u_y, H) : y \in \mathcal{Y}\}$.

For each $g \in G$, choose an element $\gamma_g \in G$ with $g\gamma_g^{-1} \in H$ and $d_{\mathcal{X}}(1, \gamma_g) = d_{\mathcal{X}}(g, H)$. Choose ξ to be a function $H \times H \to \mathcal{Y}^{\pm *}$ such that $\xi(h_1, h_2)$ represents $h_1^{-1}h_2$ in H, $|\xi(h_1, h_2)| = d_{\mathcal{Y}}(h_1, h_2)$ and $\xi(h_2, h_1) = \xi(h_1, h_2)^{-1}$. Geometrically ξ is a choice of a preferred edge path connecting each pair of vertices in the Cayley graph of H that is compatible with reversing orientation. Define a function $\Omega : G \times \mathcal{X}^{\pm 1} \to \mathcal{Y}^{\pm *}$ by $\Omega(g, x) = \xi(g\gamma_g^{-1}, gx\gamma_{gx}^{-1})$. Then $\Omega(g, x)$ represents the element $\gamma_g x \gamma_{gx}^{-1}$ of H and $\Omega(gx, x^{-1}) = \Omega(g, x)^{-1}$. Extend Ω to a function $G \times \mathcal{X}^{\pm *} \to \mathcal{Y}^{\pm *}$ by setting

$$\Omega(g, x_1 \ldots x_n) = \Omega(g, x_1)\, \Omega(gx_1, x_2)\, \ldots\, \Omega(gx_1 \ldots x_{n-1}, x_n).$$

Geometrically, we can think of Ω as a map from edge paths in the Cayley graph of G to edge paths in the Cayley graph of H which is compatible with reversing the orientation of paths.

Note that $\Omega(g, w) = \gamma_g w \gamma_{gw}^{-1}$ in G for any $w \in \mathcal{X}^{\pm *}$. Thus if w is null-homotopic then so is $\Omega(g, w)$. Given $r \in \mathbb{N}$, define $N_r = \{g \in G : d_{\mathcal{X}}(g, H) \le r\}$ and define $\mathcal{S}_1$ to be the collection of null-homotopic words $\Omega(N_{K(L)} \times \mathcal{R}^{\pm 1}) \subseteq \mathcal{Y}^{\pm *}$. We will show that $\mathcal{S}_1$ is finite by demonstrating that there is a uniform bound on the length of all words in this set. Indeed, note that, for all $g \in G$ and $x \in \mathcal{X}^{\pm 1}$, one has $d_{\mathcal{X}}(g\gamma_g^{-1}, gx\gamma_{gx}^{-1}) = d_{\mathcal{X}}(1, \gamma_g x \gamma_{gx}^{-1}) \le d_{\mathcal{X}}(g, H) + 1 + d_{\mathcal{X}}(gx, H) \le 2d_{\mathcal{X}}(g, H) + 2$. Thus $|\Omega(g, x)| = d_{\mathcal{Y}}(g\gamma_g^{-1}, gx\gamma_{gx}^{-1}) \le \Delta_H^G(2d_{\mathcal{X}}(g, H) + 2)$, where Δ_H^G is the distortion function of H in G with respect to the generating sets $\mathcal{Y}$ and $\mathcal{X}$ respectively. It follows that, for any word $w \in \mathcal{X}^{\pm *}$, one has $|\Omega(g, w)| \le |w|\Delta_H^G(2\max_{0 \le i < |w|} d_{\mathcal{X}}(gw[i], H) + 2) \le |w|\Delta_H^G(2d_{\mathcal{X}}(g, H) + 2|w| + 2)$. Thus $|s| \le R\Delta_H^G(2K(L) + 2R + 2)$ for all $s \in \mathcal{S}_1$, where $R = \max_{r \in \mathcal{R}} |r|$, and hence $\mathcal{S}_1$ is indeed finite.

If $w \in \mathcal{X}^{\pm *}$ represents an element of H then as group elements $\Omega(1, w) = \gamma_1 w \gamma_w^{-1} = w$. Thus, for each $y \in \mathcal{Y}$, we have that $\Omega(1, u_y) = y$ in H. Define $\mathcal{S}_2$ to be the collection of null-homotopic words $\{y\Omega(1, u_y)^{-1} : y \in \mathcal{Y}\} \subseteq \mathcal{Y}^{\pm *}$. We will show that H is presented by $\mathcal{Q} = \langle \mathcal{Y} \mid \mathcal{S}_1, \mathcal{S}_2 \rangle$.

Let $\sigma = y_1 \ldots y_n \in \mathcal{Y}^{\pm *}$ be an arbitrary null-homotopic word. Define σ' to be the word $u_{y_1} \ldots u_{y_n} \in \mathcal{X}^{\pm *}$. Then $\mathrm{Dep}_{\mathcal{X}}(\sigma', H) \le L$ so there exists a null $\mathcal{P}$-expression $\mathcal{E} = (x_i, r_i)_{i=1}^m$ for σ' with $\mathrm{Dep}_{\mathcal{X}}(\mathcal{E}, H) \le K(L)$. Define $\overline{\mathcal{E}}$ to be the null $\mathcal{Q}$-expression $\big(\Omega(1, x_i),\, \Omega(x_i, r_i)\big)_{i=1}^m$. The relationship between $\mathcal{E}$ and $\overline{\mathcal{E}}$ is represented schematically in Figure 2

Recall that if $g \in G$ and $x \in \mathcal{X}^{\pm 1}$ then $\Omega(g, x)^{-1} \equiv \Omega(gx, x^{-1})$. Thus if $w_1 \in \mathcal{X}^{\pm *}$ and $w_2 \in \mathcal{X}^{\pm *}$ are freely equal then for any $g \in G$ one has that $\Omega(g, w_1) \in \mathcal{Y}^{\pm *}$ and $\Omega(g, w_2) \in \mathcal{Y}^{\pm *}$ are freely equal. In particular $\Omega(1, \sigma')$ is

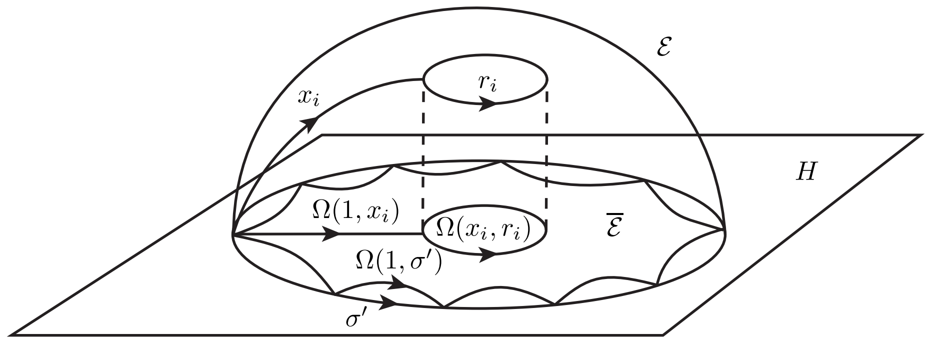

Figure 2: The relationship between $\mathcal{E}$ and $\overline{\mathcal{E}}$.

freely equal to $\Omega(1, \partial\mathcal{E})$. Also

$$\begin{aligned}\Omega(1, x_i r_i x_i) &\equiv \Omega(1, x_i)\Omega(x_i, r_i)\Omega(x_i r_i, x_i^{-1}) \\ &\equiv \Omega(1, x_i)\Omega(x_i, r_i)\Omega(x_i r_i x_i^{-1}, x_i)^{-1} \\ &\equiv \Omega(1, x_i)\Omega(x_i, r_i)\Omega(1, x_i)^{-1}\end{aligned}$$

and so $\Omega(1, \partial\mathcal{E}) \equiv \partial\overline{\mathcal{E}}$. Thus $\overline{\mathcal{E}}$ is a $\mathcal{Q}$-expression for $\Omega(1, \sigma')$.

For each $i = 1, \ldots, n$, define $\mathcal{E}_i$ to be the area 1 $\mathcal{Q}$-expression $(y_1 \ldots y_{i-1}, y_i\Omega(1, u_{y_i})^{-1})$. Then $\partial(\mathcal{E}_n \ldots \mathcal{E}_1)$ is freely equal to $y_1 \ldots y_n\Omega(1, u_{y_n})^{-1}\Omega(1, u_{y_1})^{-1} \equiv \sigma\Omega(1, \sigma')^{-1}$ and so $\mathcal{E}_n \ldots \mathcal{E}_1\overline{\mathcal{E}}$ is a $\mathcal{Q}$-expression for σ.

Now suppose that there exists a function α as in assertion (2). If we define $C = \max\{|u_y| : y \in \mathcal{Y}\}$ then $|\sigma'| \leq C|\sigma|$ and so we can choose $\mathcal{E}$ so that $\text{Area}(\mathcal{E}) \leq \alpha(C|\sigma|)$. Hence $\text{Area}_{\mathcal{Q}}(\sigma) \leq \text{Area}(\mathcal{E}_n \ldots \mathcal{E}_1\overline{\mathcal{E}}) \leq \alpha(C|\sigma|)+|\sigma|$. Define α' by $\alpha'(l) = \alpha(Cl) + l$. Then α', and hence α, is an isoperimetric function for H.

If furthermore there exists a function ρ as in assertion (3) then we can choose $\mathcal{E}$ so that additionally $\text{Rad}(\mathcal{E}) \leq \rho(C|\sigma|)$. Then $\text{Rad}(\mathcal{E}_n \ldots \mathcal{E}_1\overline{\mathcal{E}}) \leq \max\{\rho(C|\sigma|), |\sigma| - 1\} \leq \rho(C|\sigma|) + |\sigma|$. Define ρ' by $\rho'(l) = \rho(Cl) + l$. Then (α', ρ') is an area-radius pair for $\mathcal{Q}$. □

Proof of Lemma 3.18 (2). Say $\mathcal{Q} = \langle \mathcal{X} \,|\, \mathcal{R} \rangle$ and that H is finitely generated by $\mathcal{Y}$. Let $\mathcal{C}$ be a finite set of right coset representatives for H in G. For each $c \in \mathcal{C}$, choose a word $w_c \in \mathcal{X}^{\pm *}$ representing c in G. Define $L = \max_{c\in\mathcal{C}}\{|w_c|\}$. Then for each $g \in G$, there exists $c \in \mathcal{C}$ so that $gc^{-1} \in H$ and hence $d_{\mathcal{X}}(g, H) \leq L$. Thus, for any $\mathcal{Q}$-expression $\mathcal{E}$, one has that $\text{Dep}_{\mathcal{X}}(\mathcal{E}, H) \leq L$. Proposition 10.4 therefore gives a finite collection of words $\mathcal{S} \in \mathcal{Y}^{\pm *}$ and functions $\bar{\alpha}, \bar{\rho} : \mathbb{N} \to \mathbb{N}$ with $\bar{\alpha} \simeq \alpha$ and $\bar{\rho} \simeq \rho$ so that H is presented by $\bar{\mathcal{P}} = \langle \mathcal{Y} \,|\, \mathcal{S} \rangle$ and $(\bar{\alpha}, \bar{\rho})$ is an area-radius pair for $\bar{\mathcal{P}}$. The result then follows by Proposition 3.15. □

11 Full coabelian subdirect products

11.1 The main theorem

Definition 11.1. Let H be a subgroup of a group G. If $[G, G] \leq H$, then we say that H is *coabelian* in G. If there exists a finite index subgroup $G' \leq G$

so that $[G', G'] \leq H$, then we say that H is *virtually-coabelian* in G. In this situation, we define the *corank* of H in G to be $\dim\left(\frac{G'}{G' \cap H} \otimes \mathbb{Q}\right)$. Note that this is independent of the finite index subgroup $G' \leq G$ chosen.

Definition 11.2. Let H be a subgroup of a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$. If $\Gamma_i H = D$ for each i, then we say that H is *full* in D. If $[D : \Gamma_i H] < \infty$ for each H, then we say H is *virtually-full* in D. Note that these definitions are dependent upon a choice of a particular decomposition of D as a direct product.

Section 11 of this thesis is dedicated to proving the following result.

Theorem 11.3. *Let H be a virtually-full, virtually-coabelian subgroup of a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$, with corank r.*

(1) Suppose each Γ_i is finitely generated and $n \geq 2$. Then H is finitely generated and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.

(2) Suppose each Γ_i is finitely presented and $n \geq 3$. Then H is finitely presented.

(3) Suppose each Γ_i is finitely presented and $n \geq 3$. For each i, let (α_i, ρ_i) be an area-radius pair for some finite presentation of Γ_i. Define

$$\alpha(l) = \max(\{l^2\} \cup \{\alpha_i(l) \,:\, 1 \leq i \leq n\})$$

and

$$\rho(l) = \max(\{l\} \cup \{\rho_i(l) \,:\, 1 \leq i \leq n\}).$$

Then $\rho^{2r}\alpha$ is an isoperimetric function for H

(4) Suppose that each Γ_i is finitely presented and that $n \geq \max\{3, 2r\}$. Let β_1 and β_2 be the Dehn functions of some finite presentations of $\Gamma_1 \times \ldots \times \Gamma_{n-r}$ and $\Gamma_{n-r+1} \times \ldots \times \Gamma_n$ respectively. Then the function β defined by

$$\beta(l) = l\beta_1(l^2) + \beta_2(l)$$

is an isoperimetric function for H.

Note that the finite generation of the Γ_i ensures that H has finite corank in D. Furthermore, for a fixed D, the corank of a virtually-coabelian subgroup $H \leq D$ is bounded by the corank of $[D, D]$. It follows that there is a uniform polynomial isoperimetric function for all virtually-full, virtually-coabelian subgroups of D. Also observe that the finiteness properties of the Stallings-Bieri groups SB_1 and SB_2 demonstrate the necessity of the conditions $n \geq 2$ and $n \geq 3$ respectively.

11.2 Reductions of the main theorem

Proposition 11.4. *Theorem 11.3 is true if and only if it holds under the following additional hypotheses:*

(i) H is full in D.

(ii) H is coabelian in D.

(iii) D/H is finitely generated free abelian.

Note that these stronger hypotheses hold precisely when H is the kernel of a homomorphism $\theta : \Gamma_1 \times \ldots \times \Gamma_n \to \mathbb{Z}^r$ with the restriction of θ to each factor Γ_i surjective. In order to perform the reduction of Proposition 11.4 we will need the following two lemmas.

Lemma 11.5. *Let H be a virtually-full, virtually-coabelian subgroup of the direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$. Then there exists a finite index subgroup $D' \leq D$ so that $H \cap D'$ is full and coabelian in $D' = (D' \cap \Gamma_1) \times \ldots \times (D' \cap \Gamma_n)$.*

Proof. Since H is virtually-coabelian in D, there exists a finite index subgroup $\bar{D} \leq D$ with $[\bar{D}, \bar{D}] \leq H$. Define $\bar{H}$ to be the finite index subgroup $H \cap \bar{D} \leq H$, and, for each i, define $\bar{\Gamma}_i$ to be the finite index subgroup $\Gamma_i \cap \bar{D} \leq \Gamma_i$.

Since H is virtually-full in D, $[D : \Gamma_i H] < \infty$ for each i. Thus each $\bar{\Gamma}_i \bar{H}$ has finite index in $\bar{D}$, since $[\bar{D} : \bar{\Gamma}_i \bar{H}] \leq [D : \bar{\Gamma}_i \bar{H}] = [D : \Gamma_i H][\Gamma_i H : \bar{\Gamma}_i \bar{H}] < \infty$. Define D' to be the finite index subgroup $\cap_{i=1}^{n} \bar{\Gamma}_i \bar{H} \leq D$ and, for each i, define $\Gamma'_i = \bar{\Gamma}_i \cap D'$. Note that $\bar{H} \leq D'$ and hence that $\bar{H} = H \cap D'$. For each k, $\Gamma'_k \bar{H} = (\bar{\Gamma}_k \cap D')\bar{H} = \bar{\Gamma}_k \bar{H} \cap D' = D'$ and so $\bar{H} = H \cap D'$ is full in D'. Furthermore, $[D', D'] \leq [\bar{D}, \bar{D}] \leq \bar{H}$ and so $\bar{H}$ is coabelian in D'. □

Lemma 11.6. *Let G be a non-hyperbolic, finitely presented group, and let δ be the Dehn function of some finite presentation of G. Then there exists $C \in \mathbb{N}$ so that $l^2 \leq C\delta(l) + C$.*

Proof. Since G is not hyperbolic, the function δ satisfies $\delta(l) \succeq l^2$ [15, Theorem 6.1.5]. Hence there exists $K \in \mathbb{N}$ such that $l^2 \leq K\delta(Kl + K) + Kl + K$ whence $l^2 \leq K\delta(2Kl) + 2Kl$. This implies that $\frac{1}{2}l^2 + \frac{1}{2}l^2 - 2Kl \leq K\delta(2Kl)$. Note that $\frac{1}{2}l^2 - 2Kl \geq -2K^2$, so $\frac{1}{2}l^2 - 2K^2 \leq K\delta(2Kl)$, which implies that $l^2 \leq 2K\delta(2Kl) + 4K^2$. We thus have that

$$\begin{aligned} l^2 &= 4K^2(l/(2K))^2 \\ &\leq 4K^2\lfloor l/(2K)\rfloor^2 + 4K^2 \\ &\leq 4K^2(2K\delta(2K\lfloor l/(2K)\rfloor) + 4K^2) + 4K^2 \\ &\leq 8K^3\delta(l) + 16K^4 + 4K^2. \end{aligned}$$

□

Proof of Proposition 11.4. Let H be a virtually-full, virtually-coabelian subgroup of a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$ with corank r. Suppose that each Γ_i finitely generated and that Theorem 11.3 is true under the additional hypotheses (i), (ii) and (iii).

By Lemma 11.5, there exists a finite index subgroup $D' \leq D$ so that $H \cap D'$ is full and coabelian in D'. Since D is finitely generated, we may, by replacing D' by a finite index subgroup if necessary, assume that $D'/(H \cap D')$ is free abelian of rank r. Define $H' = H \cap D'$ and, for each i, define $\Gamma'_i = \Gamma_i \cap D'$. Note that $[H : H'] < \infty$ and $[\Gamma_i : \Gamma'_i] < \infty$. Thus each Γ'_i is finitely generated.

Now suppose that $n \geq 2$. Since we assumed that Part (1) of Theorem 11.3 is true under the additional hypotheses, it follows that H' is finitely generated and that the distortion function Δ' of H' in D' satisfies $\Delta'(l) \preccurlyeq l^2$. Since D' has finite index in D it is undistorted. Thus by Lemma 4.3, the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.

Now suppose that $n \geq 3$ and that each Γ_i is finitely presented. Then each Γ_i' is finitely presented. Since we assumed that Part (2) of Theorem 11.3 is true under the additional hypotheses, it follows that H', and hence H, is finitely presented. Let $\alpha, \rho, \alpha_i, \rho_i$ be as in the statement of Part (3) of Theorem 11.3. By Lemma 3.18 (2), there exists, for each i, functions $\alpha_i', \rho_i' : \mathbb{N} \to \mathbb{N}$ with $\alpha_i' \simeq \alpha_i$ and $\rho_i' \simeq \rho_i$ so that (α_i', ρ_i') is an area-radius pair for some finite presentation of Γ_i'. Define $\alpha'(l) = \max(\{l^2\} \cup \{\alpha_i'(l) : 1 \leq i \leq n\})$ and $\rho'(l) = \max(\{l\} \cup \{\rho_i'(l) : 1 \leq i \leq n\})$. Then, by the assumption that Part (3) of Theorem 11.3 is true under the additional hypotheses, ${\rho'}^{2r}\alpha'$ is an isoperimetric function for H'. By definition of the equivalence $\simeq$, there exists a constant K such that $\alpha_i'(l) \leq K\alpha_i(Kl+K)+Kl+K$ and $\rho_i'(l) \leq K\rho_i(Kl+K)+Kl+K$ for all i. Thus $\alpha'(l) \leq K\alpha(Kl+K)+Kl+K$ and $\rho'(l) \leq K\rho(Kl+K)+Kl+K$. Since $\rho(l) \geq l$ and $\alpha(l) \geq l^2 \geq l$ we have that $\alpha'(l) \leq (K+1)\alpha(Kl+K)$ and $\rho'(l) \leq (K+1)\rho(Kl+K)$. Thus $(\rho'(l))^{2r}\alpha'(l) \leq (K+1)^{2r+1}(\rho(Kl+K))^{2r}\alpha(Kl+K)$ and hence $\rho'^{2r}\alpha' \preceq \rho^{2r}\alpha$. It follows that $\rho^{2r}\alpha$ is an isoperimetric function for H' and hence, by Lemma 3.18 (1), an isoperimetric function for H.

Finally, suppose that $n \geq \max\{3, 2r\}$. Let β_1, β_2, β be as in the statement of Part (4) of Theorem 11.3. Let β_1' and β_2' be the Dehn functions of some finite presentations of $\Gamma_1' \times \ldots \times \Gamma_{n-r}'$ and $\Gamma_{n-r+1}' \times \ldots \times \Gamma_n'$ respectively and define $\beta'(l) = l\beta_1'(l^2)+\beta_2'(l)$. Then, by the assumption that Part (4) of Theorem 11.3 is true under the additional hypotheses, β' is an isoperimetric function for H'. By Lemma 3.18 (1), $\beta_1' \simeq \beta_1$ and $\beta_2' \simeq \beta_2$ and so, by the definition of $\simeq$-equivalence, there exists a constant $K \in \mathbb{N}$ so that $\beta_1'(l) \leq K\beta_1(Kl+K)+Kl+K$ and $\beta_2'(l) \leq K\beta_2(Kl+K)+Kl+K$. Then

$$\begin{aligned}\beta'(l) &\leq l[K\beta_1(Kl^2+K)+Kl^2+K]+K\beta_2(Kl+K)+Kl+K \\ &= Kl\beta_1(Kl^2+K)+K\beta_2(Kl+K)+Kl^3+2Kl+K.\end{aligned}$$

By construction, H' is the kernel of a homomorphism $\Gamma_1' \times \ldots \times \Gamma_n' \to \mathbb{Z}^r$ that is surjective on each factor Γ_i'. Theorem 11.3 (4) is trivially true when $r = 0$, so we may assume that $r \geq 1$. It follows that each Γ_i', and hence each Γ_i, contains an element of infinite order. The condition $n \geq \{3, 2r\}$ implies that $n - r \geq 2$, and so $\Gamma_1 \times \ldots \times \Gamma_{n-r}$ contains $\mathbb{Z}^2$ as a subgroup and hence is not hyperbolic. By Lemma 11.6 there thus exists $C \in \mathbb{N}$ so that $l^2 \leq C\beta_1(l)+C$. We now have

$$\begin{aligned}\beta'(l) &\leq Kl\beta_1(Kl^2+K)+K\beta_2(Kl+K)+KCl\beta_1(l)+KCl+2Kl+K \\ &\leq K(C+1)l\beta_1(Kl^2+K)+K\beta_2(Kl+K)+K(C+2)l+K \\ &\leq K(C+1)\beta(Kl+K)+K(C+2)l+K.\end{aligned}$$

Thus $\beta' \preceq \beta$ and so β is an isoperimetric function for H' and hence, by Lemma 3.18, an isoperimetric function for H. $\square$

11.3 Finite generation, distortion and finite presentation

Combined with the reduction of Proposition 11.4, the following result proves Parts (1) and (2) of Theorem 11.3.

Theorem 11.7. *Let θ be a homomorphism from a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$ of groups to a finitely generated free abelian group A such that, for each i, the restriction of θ to Γ_i is surjective.*

(1) If each Γ_i is finitely generated and $n \geq 2$ then $\ker\theta$ is finitely generated and the distortion function Δ of $\ker\theta$ in D satisfies $\Delta(l) \preccurlyeq l^2$.

(2) If each Γ_i is finitely presented and $n \geq 3$ then $\ker\theta$ is finitely presented.

Proof. For Part (1), observe that $\ker\theta$ is the fibre product of the homomorphisms $\theta|_{\Gamma_1}$ and $-\theta|_{\Gamma_2\times\ldots\times\Gamma_n}$. Since A is finitely generated free abelian, it admits a quadratic isoperimetric function. Thus, by Lemma 9.5, $\ker\theta$ is finitely generated and $\Delta(l) \preceq l^2$. Hence, by definition of the relation $\preceq$, there exists $C \in \mathbb{N}$ so that $\Delta(l) \leq C(Cl + C)^2 + Cl + C \leq (3C^3 + C)l^2 + C^3 + C$. Thus $\Delta(l) \preccurlyeq l^2$.

For Part (2), observe that $\ker\theta$ is the fibre product of the homomorphisms $p_1 := \theta|_{\Gamma_1\times\Gamma_2}$ and $p_2 := -\theta|_{\Gamma_3\times\ldots\times\Gamma_n}$. Since $\ker p_1$ is the fibre product of the homomorphisms $\theta|_{\Gamma_1}$ and $-\theta|_{\Gamma_2}$, it is finitely generated by Lemma 9.5. Thus $\ker\theta$ is finitely presented by Theorem 9.4. $\square$

11.4 Heights

Throughout the remainder of Section 11, we will be considering a homomorphism θ from a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$ of groups to a finitely generated free abelian group A such that the restriction of θ to each Γ_i is surjective. After establishing some notation, which will be maintained throughout Sections 11.4–11.6, we will define certain *height functions* that measure the departure from $\ker\theta$ of words, expressions and sequences in each of the r directions given by the $\mathbb{Z}$-factors of $A \cong \mathbb{Z}^r$.

Let $t_1, \ldots, t_r$ be a free abelian basis for A. For each i, let $\mathcal{A}_i = \{a_1^{(i)}, \ldots, a_r^{(i)}\} \subseteq \Gamma_i$ be a collection of elements with $\theta(a_k^{(i)}) = t_k$, and let $\mathcal{B}_i \subseteq \Gamma_i$ be a collection of elements with $\theta(\mathcal{B}_i) = \{1\}$ and so that $\mathcal{X}_i := \mathcal{A}_i \cup \mathcal{B}_i$ generates Γ_i. Define $\mathcal{X}$ to be the generating set $\cup_{i=1}^n \mathcal{X}_i$ for D.

For each $i = 1, \ldots, r$, define $\mathbb{Z}^{(i)}$ to be the quotient of A by the subgroup generated by $\{t_j : j \neq i\}$. Define $\theta_i : D \to \mathbb{Z}$ to be the composition of θ with the quotient homomorphism $A \twoheadrightarrow \mathbb{Z}^{(i)}$. Abusing notation, we will also write θ_i for the map $\mathcal{X}^{\pm *} \times \mathbb{N} \to \mathbb{Z}$ given by $\theta_i(w, l) = \theta_i(w[l])$.

For each $i = 1, \ldots, r$, we define the *i-height* of a word to be the departure of the word in the $\mathbb{Z}^{(i)}$-direction, as measured by θ_i. Specifically, given $w \in \mathcal{X}^{\pm *}$, define

$$\mathrm{height}_i(w) = \max_{0 \leq j \leq |w|} \{|\theta_i(w, j)|\}.$$

If $\mathcal{P} = \langle \mathcal{X} \,|\, \mathcal{S} \rangle$ is a presentation for D, then the i-heights of $\mathcal{P}$-sequences and $\mathcal{P}$-expressions are defined similarly. Given a $\mathcal{P}$-expression $\mathcal{E} = (x_j, s_j)_{j=1}^m$ and a $\mathcal{P}$-sequence $\Sigma = (\sigma_j)_{j=0}^m$ define

$$\mathrm{height}_i(\mathcal{E}) = \max_{1 \leq j \leq m} \{\mathrm{height}_i(x_j)\},$$
$$\mathrm{height}_i(\Sigma) = \max_{0 \leq j \leq m} \{\mathrm{height}_i(\sigma_j)\}.$$

The following lemma makes precise the relationship between heights and departures.

Lemma 11.8.

$$\operatorname{height}_i(w) \leq \operatorname{Dep}_{\mathcal{X}}(w, \ker\theta) \leq \sum_{j=1}^{r} \operatorname{height}_j(w)$$

$$\operatorname{height}_i(\mathcal{E}) \leq \operatorname{Dep}_{\mathcal{X}}(\mathcal{E}, \ker\theta) \leq \sum_{j=1}^{r} \operatorname{height}_j(\mathcal{E})$$

$$\operatorname{height}_i(\Sigma) \leq \operatorname{Dep}_{\mathcal{X}}(\Sigma, \ker\theta) \leq \sum_{j=1}^{r} \operatorname{height}_j(\Sigma)$$

Proof. For any prefix u of w, there exists a word $v \in \mathcal{X}^{\pm *}$ with $|v| \leq \operatorname{Dep}_{\mathcal{X}}(w, \ker\theta)$ such that $uv^{-1} \in \ker\theta$. Then $|\theta_i(u)| = |\theta_i(v)| \leq |v|$ and so $\operatorname{height}_i(w) \leq \operatorname{Dep}_{\mathcal{X}}(w, \ker\theta)$.

For any word $u \in \mathcal{X}^{\pm *}$, the word $u(a_1^{(1)})^{-\theta_1(u)} \dots (a_r^{(1)})^{-\theta_r(u)}$ represents an element of the kernel, so $\mathrm{d}_{\mathcal{X}}(u, \ker\theta) \leq \sum_{j=1}^{r} |\theta_j(u)|$. Taking the maximum over all prefixes u of w gives the inequality $\operatorname{Dep}_{\mathcal{X}}(w, \ker\theta) \leq \sum_{j=1}^{m} \operatorname{height}_j(w)$.

Maximising over all the words x_j or all the words σ_j gives the other inequalities. $\square$

The following lemma asserts that in order to produce a $\mathcal{P}$-expression for a word w with some bounds on its area and heights, it suffices to produce a null $\mathcal{P}$-sequence for w satisfying the given bounds.

Lemma 11.9. *Let Σ be a $\mathcal{P}$-sequence converting τ to τ'. Then there exists a $\mathcal{P}$-expression $\mathcal{E}$ for $\tau(\tau')^{-1}$ with* $\operatorname{Area}(\mathcal{E}) = \operatorname{Area}(\Sigma)$ *and* $\operatorname{height}_i(\mathcal{E}) \leq \operatorname{height}_i(\Sigma)$ *for each i.*

Proof. Say $\Sigma = (\sigma_j)_{j=0}^{m}$, where $\sigma_0 \equiv \tau$ and $\sigma_m \equiv \tau'$. Define Σ_1 to be the $\mathcal{P}$-sequence $(\sigma_j)_{j=0}^{m-1}$. By induction, there exists a $\mathcal{P}$-expression $\mathcal{E}_1$ for $\sigma_0(\sigma_{m-1})^{-1}$ with $\operatorname{Area}(\mathcal{E}_1) = \operatorname{Area}(\Sigma_1)$ and $\operatorname{height}_i(\mathcal{E}_1) \leq \operatorname{height}_i(\Sigma_1) \leq \operatorname{height}_i(\Sigma)$ for each i. If σ_m is obtained from σ_{m-1} by a free expansion or reduction then $\sigma_0{\sigma_m}^{-1} \overset{\mathrm{fr}}{=} \sigma_0{\sigma_{m-1}}^{-1}$ and the result follows on taking $\mathcal{E} = \mathcal{E}_1$. The other possibility is that σ_m is obtained from σ_{m-1} by an application-of-a-relator move. Then $\sigma_{m-1} \equiv \alpha u \beta$ and $\sigma_m \equiv \alpha v \beta$ where uv^{-1} is a cyclic conjugate of a relator $s \in \mathcal{S}^{\pm 1}$ and α and β are some words in $\mathcal{X}^{\pm *}$. Observe that either $uv^{-1} \overset{\mathrm{fr}}{=} u' s u'^{-1}$ where u' is a prefix of u or else $uv^{-1} \overset{\mathrm{fr}}{=} v' s v'^{-1}$ where v' is a prefix of v.

In the first case, we have that $\sigma_{m-1}\sigma_m^{-1} \overset{\mathrm{fr}}{=} \alpha u v^{-1} \alpha^{-1} \overset{\mathrm{fr}}{=} \alpha u' s (\alpha u')^{-1}$. Note that $\operatorname{height}_i(\alpha u') \leq \operatorname{height}_i(\Sigma)$ since $\alpha u'$ is a prefix of σ_{m-1}. If we take $\mathcal{E}_2 = (\alpha u', s)$ then $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$ has the required properties. In the second case we can take $\mathcal{E}_2 = (\alpha v', s)$ and the result follows similarly. $\square$

When considering the areas of sequences, one only has to take account of the application-of-a-relator moves, and can ignore the free expansions and contractions. The following lemma shows that the same is true when one is considering the heights of sequences.

Lemma 11.10. *Let $w_1, w_2 \in \mathcal{X}^{\pm *}$ be freely equal words. Then there exists a $\mathcal{P}$-sequence Σ converting w_1 to w_2 with* $\operatorname{Area}(\Sigma) = 0$ *and* $\operatorname{height}_i(\Sigma) \leq \max\{\operatorname{height}_i(w_1), \operatorname{height}_i(w_2)\}$ *for each i.*

Proof. Let $\bar{w}$ be the unique freely reduced word in the free equivalence class of w_1 and w_2. For each $k = 1, 2$, let $\Sigma_k = \left(\sigma_j^{(k)}\right)_{j=0}^{m_k}$ be a $\mathcal{P}$-sequence converting w_k to $\bar{w}$ where each $\sigma_{j+1}^{(k)}$ is obtained from $\sigma_j^{(k)}$ by a free reduction. Then $\text{height}_i(\sigma_{j+1}^{(k)}) \leq \text{height}_i(\sigma_j^{(k)})$ and so $\text{height}_i(\Sigma_k) \leq \text{height}_i(w_k)$. Define Σ_2' to be the $\mathcal{P}$-sequence $\sigma_{m_2}^{(2)}, \sigma_{m_2-1}^{(2)}, \ldots, \sigma_0^{(2)}$ converting $\bar{w}$ to w_2. Then $\Sigma = \Sigma_1 \Sigma_2'$ has the required properties. $\square$

The area of a null-homotopic word is equal to the area of its inverse. The following lemma is the analogous result for heights.

Lemma 11.11. *Let Σ be a null $\mathcal{P}$-sequence for the word $w \in \mathcal{X}^{\pm *}$. Then there exists a null $\mathcal{P}$-sequence Σ' for w^{-1} with* $\text{Area}(\Sigma') = \text{Area}(\Sigma)$ *and* $\text{height}_k(\Sigma') = \text{height}_k(\Sigma)$ *for each k.*

Proof. For any null-homotopic word $u \in \mathcal{X}^{\pm *}$, one has that $\theta_k(u) = 0$ and hence $\text{height}_k(u) = \text{height}_k(u^{-1})$. Thus if $\Sigma = (\sigma_i)_{i=0}^m$ then we can take Σ' to be $(\sigma_i^{-1})_{i=0}^m$. $\square$

Throughout Section 11 we will frequently wish to assert that there exists a $\mathcal{P}$-sequence for a word with some stated bounds on its area and heights. However, for reasons of space and readability we wish to avoid having to present all of the data required to define a particular such sequence. We therefore redefine the notion of a $\mathcal{P}$-scheme to additionally take account of heights. Thus, throughout this section, a $\mathcal{P}$-scheme is defined to consist of a sequence $(\sigma_i)_{i=1}^m$ of words in $\mathcal{X}^{\pm *}$ and sequences $\left(\alpha_i\right)_{i=1}^{m-1}, \left(h_i^{(1)}\right)_{i=1}^{m-1}, \ldots, \left(h_i^{(r)}\right)_{i=1}^{m-1}$ of integers so that, for each i, there exists a $\mathcal{P}$-sequence Σ_i converting σ_i to σ_{i+1} with $\text{Area}(\Sigma) \leq \alpha_i$ and $\text{height}_k(\Sigma) \leq h_i^{(k)}$ for each k. The notion of a null $\mathcal{P}$-scheme is redefined similarly.

11.5 Distortion

In this section we prove the following result.

Theorem 11.12. *Let θ be a homomorphism from a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$ of $n \geq 2$ finitely generated groups to a finitely generated free abelian group A such that the restriction of θ to each Γ_i is surjective. Then* $\ker\theta$ *is finitely generated and the distortion function Δ of* $\ker\theta$ *in D satisfies $\Delta(l) \preccurlyeq l^{r+1}$, where $r = \dim A \otimes \mathbb{Q}$.*

Note that when combined with Proposition 11.4, this result provides an alternative proof of the assertion of finite generation in Part (1) of Theorem 11.3. However, the main purpose of Theorem 11.12 is to act as a warm up for the proof of Theorem 11.15, which is analogous but more involved.

We continue with the notation of the previous section. Recall that $t_1, \ldots, t_r$ is a free abelian basis for A. For each i, $\mathcal{X}_i = \mathcal{A}_i \cup \mathcal{B}_i$ is a generating set for Γ_i, with $\mathcal{A}_i = \{a_1^{(i)}, \ldots, a_r^{(i)}\}$ satisfying $\theta(a_k^{(i)}) = t_k$ and with $\theta(\mathcal{B}_i) = \{1\}$. Thus D is generated by $\mathcal{X} = \cup_{i=1}^n \mathcal{X}_i$. For each $i = 1, \ldots, r$, $\mathbb{Z}^{(i)}$ is the infinite cyclic subgroup of A generated by t_i, and $\theta_i : D \to \mathbb{Z}$ is the composition of θ with the projection homomorphism $A \twoheadrightarrow \mathbb{Z}^{(i)}$. We also write θ_i for the map $\mathcal{X}^{\pm *} \times \mathbb{N} \to \mathbb{Z}$ given by $\theta_i(w, l) = \theta_i(w[l])$.

The proof of Theorem 11.12 makes use of Proposition 10.2: we show that every element $g \in \ker\theta$ can be represented by a word in $\mathcal{X}^{\pm *}$ which has uniformly bounded departure. We first represent g by an arbitrary geodesic word, representing an edge path in the Cayley graph of D, which we then 'pull down' until it lies close to the kernel. Recall the height functions, defined in Section 11.4, which measure departure in each of the r different directions given by the $\mathbb{Z}$-factors of A. Proposition 11.13 shows that it is possible to pull down a word in a particular direction without increasing its height in the other directions. The trade off to this process is that the length of the word is increased. In Proposition 11.14 we show that, by applying Proposition 11.13 repeatedly, an arbitrary word can be pulled down to a word which has small height in every direction. This word thus has small departure from the kernel.

For each $k = 1, \ldots, r$, we will define a function Φ_k that will be used to pull down words in the k^{th} direction. The idea is that if $w \in \mathcal{X}^{\pm *}$ represents an element of $\ker\theta$ then $\Phi_k(w)$ will represent the same element as w but will have $\text{height}_k(\Phi_k(w)) \le 1$. In actual fact, we will find it useful to define Φ_k to be a function $\mathcal{X}^{\pm *} \times \mathbb{Z} \to \mathcal{X}^{\pm *}$, with $\Phi_k(w, h)$ representing the element $\left(a_k^{(1)}\right)^h w \left(a_k^{(1)}\right)^{-h-\theta_k(w)} \in \ker\theta$. Geometrically, one thinks of the input to Φ_k as being an edge path in the Cayley graph of G which starts at height h and is labelled by the word w. This pulling down process is represented schematically in Figure 3.

The reader should note that Φ_k is only defined when $n \ge 2$, and from now on we assume that this is the case. For brevity write e_k for $a_k^{(1)}$ and f_k for $a_k^{(2)}$. Define Φ_k on $\mathcal{X}^{\pm 1} \times \mathbb{Z}$ by

$$\Phi_k(x, h) \equiv \begin{cases} (e_k {f_k}^{-1})^h x {f_k}^{-\theta_k(x)} (e_k {f_k}^{-1})^{-h-\theta_k(x)} & \text{if } x \in \mathcal{X}_1, \\ x {e_k}^{-\theta_k(x)} & \text{if } x \in \mathcal{X}_2 \cup \ldots \cup \mathcal{X}_n \end{cases}$$

and

$$\Phi_k(x^{-1}, h) \equiv \begin{cases} (e_k {f_k}^{-1})^h {f_k}^{\theta_k(x)} x^{-1} (e_k {f_k}^{-1})^{-h+\theta_k(x)} & \text{if } x \in \mathcal{X}_1, \\ {e_k}^{\theta_k(x)} x^{-1} & \text{if } x \in \mathcal{X}_2 \cup \ldots \cup \mathcal{X}_n. \end{cases}$$

Extend Φ_k over $\mathcal{X}^{\pm *} \times \mathbb{Z}$ by setting

$$\Phi_k(w, h) \equiv \prod_{j=1}^{|w|} \Phi_k(w(j), \theta_k(w, j-1) + h).$$

Proposition 11.13. *Let $w, w' \in \mathcal{X}^{\pm *}$, $h \in \mathbb{Z}$ and $k \in \{1, \ldots, r\}$. Then Φ_k enjoys the following properties:*

(1) $\Phi_k(w, h) = {e_k}^h w {e_k}^{-h-\theta_k(w)}$ *in D.*

(2) $|\Phi_k(w, h)| \le 4|w|(\text{height}_k(w) + |h| + 1)$.

(3) $\text{height}_i(\Phi_k(w, h)) \le \begin{cases} 1 & \textit{if } i = k, \\ \text{height}_i(w) & \textit{if } i \ne k. \end{cases}$

(4) $\Phi_k(w, h)^{-1} \equiv \Phi_k(w^{-1}, \theta_k(w) + h)$.

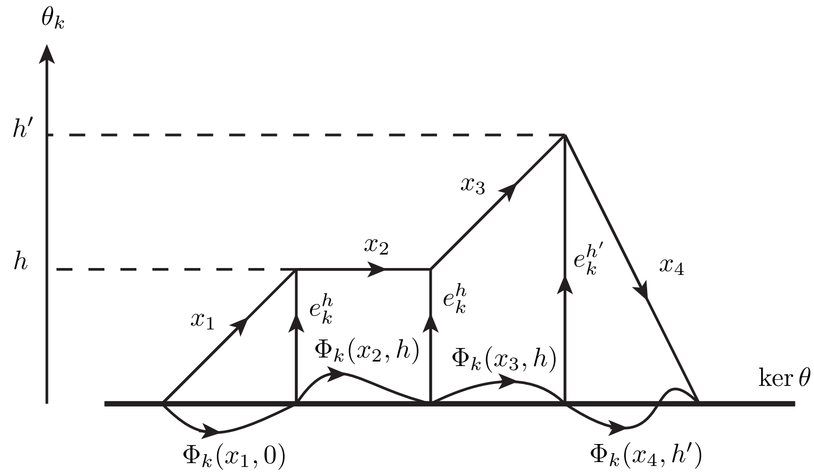

Figure 3: Pulling down a word $w \equiv x_1x_2x_3x_4$.

(5) $\Phi_k(ww', h) \equiv \Phi_k(w, h)\Phi_k(w', \theta_k(w) + h)$.

(6) If $w \overset{fr}{=} w'$ *then* $\Phi_k(w, h) \overset{fr}{=} \Phi_k(w', h)$.

Proof.

(1) If $w \in \mathcal{X}^{\pm 1}$ then one checks directly that property (1) holds. Thus for an arbitrary $w \in \mathcal{X}^{\pm *}$

$$\begin{aligned}
\Phi_k(w, h) &\overset{D}{=} \prod_{j=1}^{|w|} {e_k}^{\theta_k(w,j-1)+h} w(j) {e_k}^{-\theta_k(w,j-1)-h-\theta_k(w(j))} \\
&\equiv \prod_{j=1}^{|w|} {e_k}^{\theta_k(w,j-1)+h} w(j) {e_k}^{-\theta_k(w,j)-h} \\
&\overset{\text{fr}}{=} {e_k}^{\theta_k(w,0)+h} \left(\prod_{j=1}^{|w|} w(j) \right) {e_k}^{-\theta_k(w,m)-h} \\
&\equiv {e_k}^{h} w {e_k}^{-h-\theta_k(w)}.
\end{aligned}$$

(2) If $x \in \mathcal{X}^{\pm 1}$ then $|\Phi_k(x, h)| \leq 4(|h| + 1)$. Thus

$$\begin{aligned}
|\Phi_k(w, h)| &\leq |w| \max_{1 \leq j \leq |w|} \left\{ |\Phi_k(w(j), \theta_k(w, j-1) + h)| \right\} \\
&\leq |w| \max_{1 \leq j \leq |w|} \left\{ 4(|\theta_k(w, j-1)| + |h| + 1) \right\} \\
&\leq 4|w|(\text{height}_k(w) + |h| + 1).
\end{aligned}$$

(3) If $x \in \mathcal{X}^{\pm 1}$ and u is a prefix of $\Phi_k(x, h)$ then

$$\theta_i(u) \in \begin{cases} \{0, 1\} & \text{if } i = k, \\ \{0, \theta_i(x)\} & \text{if } i \neq k. \end{cases}$$

Furthermore

$$\theta_i(\Phi_k(x,h)) = \begin{cases} 0 & \text{if } i = k, \\ \theta_i(x) & \text{if } i \neq k. \end{cases}$$

Thus if v is a prefix of $\Phi_k(w,h)$ then $\theta_k(v) \in \{-1,0,1\}$ and so $\mathrm{height}_k(\Phi_k(w,h)) \leq 1$. If $i \neq k$ then $\theta_i(v) = \theta_i(v')$ for some prefix v' of w. Thus $\mathrm{height}_i(\Phi_k(w,h)) \leq \mathrm{height}_i(w)$.

(4) One checks directly that if $x \in \mathcal{X}^{\pm 1}$ then $\Phi_k(x^{-1},h) \equiv \Phi_k(x, h - \theta_k(x))^{-1}$. Thus

$$\begin{aligned}
&\Phi_k\big(w^{-1}, \theta_k(w) + h\big) \\
&\quad\equiv \prod_{j=1}^{|w|} \Phi_k\big(w^{-1}(j), \theta_k(w^{-1}, j-1) + \theta_k(w) + h\big) \\
&\quad\equiv \prod_{j=1}^{|w|} \Phi_k\big(w(|w|-j+1)^{-1}, \theta_k(w, |w|-j+1) + h\big) \\
&\quad\equiv \prod_{j=1}^{|w|} \Phi_k\big(w(|w|-j+1), \theta_k(w, |w|-j+1) + h - \theta_k(w(|w|-j+1))\big)^{-1} \\
&\quad\equiv \prod_{j=1}^{|w|} \Phi_k\big(w(|w|-j+1), \theta_k(w, |w|-j) + h\big)^{-1} \\
&\quad\equiv \left(\prod_{l=1}^{|w|} \Phi_k\big(w(l), \theta_k(w, l-1) + h\big) \right)^{-1} \\
&\quad\equiv \Phi_k\big(w,h\big)^{-1}
\end{aligned}$$

(5)

$$\begin{aligned}
\Phi_k(ww', h) &\equiv \prod_{j=1}^{|ww'|} \Phi_k((ww')(j), \theta_k(ww', j-1) + h) \\
&\equiv \left(\prod_{j=1}^{|w|} \Phi_k(w(j), \theta_k(w, j-1) + h) \right) \\
&\qquad \left(\prod_{j=1}^{|w'|} \Phi_k(w'(j), \theta_k(w', j-1) + \theta_k(w) + h) \right) \\
&\equiv \Phi_k(w,h)\Phi_k(w', \theta_k(w) + h)
\end{aligned}$$

(6) It suffices to consider the case where w' is obtained from w by a free expansion. Say $w \equiv uv$ and $w' \equiv uxx^{-1}v$ where $u, v \in \mathcal{X}^{\pm *}$ and $x \in \mathcal{X}^{\pm 1}$.

Then

$$\begin{aligned}
&\Phi_k(w', h) \\
&\quad \equiv \Phi_k(u, h)\ \Phi_k(x, \theta_k(u) + h)\ \Phi_k(x^{-1}, \theta_k(ux) + h)\ \Phi_k(v, \theta_k(uxx^{-1}) + h) \\
&\quad \equiv \Phi_k(u, h)\ \Phi_k(x, \theta_k(u) + h)\ \Phi_k(x, \theta_k(ux) + h + \theta_k(x^{-1}))^{-1}\ \Phi_k(v, \theta_k(u) + h) \\
&\quad \equiv \Phi_k(u, h)\ \Phi_k(x, \theta_k(u) + h)\ \Phi_k(x, \theta_k(u) + h)^{-1}\ \Phi_k(v, \theta_k(u) + h) \\
&\quad \overset{\text{fr}}{=} \Phi_k(u, h)\ \Phi_k(v, \theta_k(u) + h) \\
&\quad \equiv \Phi_k(w, h)
\end{aligned}$$

□

Proposition 11.14. *Suppose $n \geq 2$. Then for all words $w \in \mathcal{X}^{\pm *}$ with $\theta(w) = 1$, there exists a word $w' \in \mathcal{X}^{\pm *}$ with the following properties:*

(1) $w' = w$ *in* D.

(2) $|w'| \leq 8^r |w|^{r+1}$.

(3) $\mathrm{height}_i(w') \leq 1$ *for all* i.

Proof. If $w \equiv \emptyset$ then the result is trivial. We may thus assume that $|w| \geq 1$. We claim that for all $j \in \{0, \ldots, r\}$ there exists a word $w_j \in \mathcal{X}^{\pm *}$ with the following properties:

(i) $w_j = w$ in D.

(ii) $|w_j| \leq 8^j |w|^{j+1}$.

(iii) $\mathrm{height}_l(w_j) \leq \begin{cases} 1 & \text{if } 1 \leq l \leq j, \\ |w| & \text{if } j+1 \leq l \leq r. \end{cases}$

The proposition then follows by taking $j = r$. We prove the claim by induction on j, with $w_0 \equiv w$. Suppose that for some j there exists a w_j with the given properties. Then define $w_{j+1} \equiv \Phi_{j+1}(w_j, 0)$. It is immediate by Proposition 11.13 (1) and (3) that w_{j+1} satisfies (i) and (iii). Furthermore, by Proposition 11.13 (2),

$$\begin{aligned}
|w_{j+1}| &\leq 4|w_j|(\mathrm{height}_{j+1}(w_j) + 1) \\
&\leq 4 \cdot 8^j |w|^{j+1}(|w| + 1) \\
&\leq 8^{j+1}|w|^{j+2}.
\end{aligned}$$

□

Proof of Theorem 11.12. Since each Γ_i is finitely generated we can take each $\mathcal{B}_i$ to be finite and so D is finitely generated by $\mathcal{X}$.

Let g be an arbitrary element of $\ker\theta$ and choose a geodesic word $w \in \mathcal{X}^{\pm *}$ representing g in D. Let $w' \in \mathcal{X}^{\pm *}$ be a word equal to w in D and satisfying properties (2) and (3) of Proposition 11.14. Then $\mathrm{Dep}_{\mathcal{X}}(w', \ker\theta) \leq r$ by Lemma 11.8 and $|w'| \leq 8^r|w|^{r+1} = 8^r(d_{\mathcal{X}}(1, g))^{r+1}$ since w is geodesic. The result follows by applying Proposition 10.2. □

11.6 Isoperimetric functions 1

In this section we prove the following result, which, when combined with Proposition 11.4, gives Parts (2) and (3) of Theorem 11.3. Note that this provides an alternative proof, in addition to that given in Theorem 11.7, of the finite presentability $\ker\theta$.

Theorem 11.15. *Let θ be a homomorphism from a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$ of $n \geq 3$ finitely presented groups to a finitely generated free abelian group A such that the restriction of θ to each Γ_i is surjective. Then $\ker\theta$ is finitely presented.*

Suppose additionally that, for each i, (α_i, ρ_i) is an area-radius pair for some finite presentation of Γ_i. Then $\rho^{2r}\alpha$ is an isoperimetric function for $\ker\theta$, where $r = \dim A \otimes \mathbb{Q}$ and α and ρ are given by

$$\alpha(l) = \max(\{l^2\} \cup \{\alpha_i(l) : 1 \leq i \leq n\})$$

and

$$\rho(l) = \max(\{l\} \cup \{\rho_i(l) : 1 \leq i \leq n\}).$$

The proof of Theorem 11.15 is analogous to the proof of Theorem 11.12, except that instead of pulling down words (representing edge paths in the Cayley graph of D) one pulls down $\mathcal{P}$-expressions (representing filling discs in the Cayley 2-complex of D). We first establish some notation.

Recall that, for each i, $\mathcal{X}_i = \mathcal{A}_i \cup \mathcal{B}_i$ is a generating set for Γ_i, with $\mathcal{A}_i = \{a_1^{(i)}, \ldots, a_r^{(i)}\}$ satisfying $\theta(a_k^{(i)}) = t_k$ and with $\theta(\mathcal{B}_i) = \{1\}$. Since each Γ_i is finitely generated we may take each $\mathcal{B}_i$ to be finite. Thus D is finitely generated by $\mathcal{X} = \cup_{i=1}^n \mathcal{X}_i$. For each i, let $\mathcal{P}_i = \langle \mathcal{X}_i \,|\, \mathcal{R}_i \rangle$ be a finite presentation for Γ_i. Define $\mathcal{R} = \cup_{i=1}^n \mathcal{R}_i$ and define $\mathcal{C}$ to be the set of relators $\{[x, y] : x \in \mathcal{X}_i, y \in \mathcal{X}_j, 1 \leq i < j \leq n\} \subseteq \mathcal{X}^{\pm *}$. Then D is finitely presented by $\mathcal{P} = \langle \mathcal{X} \,|\, \mathcal{C}, \mathcal{R} \rangle$.

The structure of the proof is as follows. Given a null-homotopic word $w \in \mathcal{X}^{\pm *}$, we will apply Proposition 11.22 to give a $\mathcal{P}$-expression for w whose area and heights (as defined in Section 11.4) are bounded in terms of α and ρ. We then pull this down to give an 'almost flat' $\mathcal{P}$-expression for w, i.e. one which has all its heights small, in the sense of being bounded in terms of the heights of w. The departure of this $\mathcal{P}$-expression is then bounded in terms of the departure of w and so the result will follow by Proposition 10.4.

As in the 1-dimensional case, we will use the functions Φ_i to successively pull down expressions in each of the r different directions. Proposition 11.20 asserts that it is possible to pull down a $\mathcal{P}$-expression in a particular direction without (essentially) increasing the heights in the other directions. In Proposition 11.21 we apply this result repeatedly to show that an arbitrary $\mathcal{P}$-expression can be pulled down to one that is almost flat.

Lemmas 11.16–11.19 give various calculations required in the proof of Proposition 11.20. When an expression for a word w is pulled down in the i^{th} direction, one does not immediately obtain an expression for w, but in fact an expression for $\Phi_i(w, 0)$. The point is that if w is almost flat, then $\Phi_i(w, 0)$ will lie close to w and so one can be converted to the other at low cost. This calculation is performed in Lemmas 11.16 and 11.17.

In order to pull down an expression $\mathcal{E}$ in the i^{th} direction, one needs almost flat fillings for the words $\Phi_i(s, h)$, where $s \in \mathcal{C}^{\pm 1} \cup \mathcal{R}^{\pm 1}$. These are provided by Lemmas 11.18 and 11.19.

Lemma 11.16. *Suppose $n \geq 2$. Then for all $x \in \mathcal{X}^{\pm 1}$, $h \in \mathbb{Z}$ and $k \in \{1, \ldots, r\}$ there exists a $\mathcal{P}$-sequence Σ converting $\Phi_k(x,h)$ to ${e_k}^h x {e_k}^{-h-\theta_k(x)}$ with*

$$\mathrm{Area}(\Sigma) \leq 2(|h|+1)^2$$

$$\mathrm{height}_i(\Sigma) \leq \begin{cases} |h|+1 & \text{if } i = k, \\ 1 & \text{if } i \neq k. \end{cases}$$

Proof. We consider 4 separate cases.

Case 1. $x \in \mathcal{X}_1$.

The following table presents a $\mathcal{P}$-scheme converting the word $\Phi_k(x,h)$ to the word ${e_k}^h x {e_k}^{-h-\theta_k(x)}$. In lines 3 and 5 we have applied Lemma 11.10.

j	σ_j	Area	height_i $(i \neq k)$	height_k
1	$(e_k f_k)^h x {f_k}^{-\theta_k(x)} (e_k {f_k}^{-1})^{-h-\theta_k(x)}$	$\frac{1}{2}\|h\|(\|h\|+1)$	1	$\max\{\|h\|, 1\}$
2	${e_k}^h {f_k}^{-h} x {f_k}^{-\theta_k(x)} (e_k {f_k}^{-1})^{-h-\theta_k(x)}$	$\frac{1}{2}(\|h\|+1)(\|h\|+2)$	1	$\|h\|+1$
3	${e_k}^h {f_k}^{-h} x {f_k}^{-\theta_k(x)} {f_k}^{h+\theta_k(x)} {e_k}^{-h-\theta_k(x)}$	0	1	$\|h\|+1$
4	${e_k}^h {f_k}^{-h} x {f_k}^h {e_k}^{-h-\theta_k(x)}$	$\|h\|$	1	$\|h\|+1$
5	${e_k}^h {f_k}^{-h} {f_k}^h x {e_k}^{-h-\theta_k(x)}$	0	1	$\|h\|+1$
6	${e_k}^h x {e_k}^{-h-\theta_k(x)}$			

Case 2. $x \in \mathcal{X}_2 \cup \ldots \cup \mathcal{X}_n$.

The following table presents a $\mathcal{P}$-scheme converting the word $\Phi_k(x,h)$ to the word ${e_k}^h x {e_k}^{-h-\theta_k(x)}$. In lines 1 and 3 we have applied Lemma 11.10.

j	σ_j	Area	height_i $(i \neq k)$	height_k
1	$x {e_k}^{-\theta_k(x)}$	0	1	$\max\{\|h\|, 1\}$
2	${e_k}^h {e_k}^{-h} x {e_k}^{-\theta_k(x)}$	$\|h\|$	1	$\|h\|+1$
3	${e_k}^h x {e_k}^{-h} {e_k}^{-\theta_k(x)}$	0	1	$\|h\|+1$
4	${e_k}^h x {e_k}^{-h-\theta_k(x)}$			

Case 3. $x \in \mathcal{X}_1^{-1}$.

Similar to the case $x \in \mathcal{X}_1$.

Case 4. $x \in \mathcal{X}_2^{-1} \cup \ldots \cup \mathcal{X}_n^{-1}$

Similar to the case $x \in \mathcal{X}_2 \cup \ldots \cup \mathcal{X}_n$. □

Lemma 11.17. *Suppose $n \geq 2$. Let $w \in \mathcal{X}^{\pm *}$, $h \in \mathbb{Z}$ and $k \in \{1, \ldots, r\}$. Then there exists a $\mathcal{P}$-sequence Σ converting $\Phi_k(w,h)$ to ${e_k}^h w {e_k}^{-h-\theta_k(w)}$ with*

$$\mathrm{Area}(\Sigma) \leq 2|w|(\mathrm{height}_k(w) + |h| + 1)^2$$

$$\mathrm{height}_i(\Sigma) \leq \begin{cases} \mathrm{height}_k(w) + |h| + 1 & \text{if } i = k, \\ \mathrm{height}_i(w) + 1 & \text{if } i \neq k. \end{cases}$$

Proof. Define

$$\sigma_1 \equiv \prod_{j=1}^{|w|} \Phi_k(w(j), \theta_k(w, j-1) + h),$$

$$\sigma_2 \equiv \prod_{j=1}^{|w|} {e_k}^{h+\theta_k(w,j-1)} w(j) {e_k}^{-h-\theta_k(w,j)},$$

$$\sigma_3 \equiv {e_k}^{h} w {e_k}^{-h-\theta_k(w)}.$$

By Lemma 11.16, there exists a $\mathcal{P}$-sequence Σ_1 converting σ_1 to σ_2 with $\mathrm{Area}(\Sigma_1) \le 2|w| \max_{1\le j\le |w|}(|\theta_k(w,j-1)+h|+1)^2 \le 2|w|(\mathrm{height}_k(w)+|h|+1)^2$ and

$$\mathrm{height}_i(\Sigma_1) \le \begin{cases} \mathrm{height}_k(w) + |h| + 1 & i = k, \\ 1 & i \ne k. \end{cases}$$

By Lemma 11.10, there exists a $\mathcal{P}$-sequence Σ_2 converting σ_2 to σ_3 with $\mathrm{Area}(\Sigma_2) = 0$ and

$$\mathrm{height}_i(\Sigma_2) \le \begin{cases} \mathrm{height}_k(w) + |h| & i = k, \\ \mathrm{height}_i(w) & i \ne k. \end{cases}$$

Take $\Sigma = \Sigma_1\Sigma_2$. □

Lemma 11.18. *Suppose $n \ge 2$. Then there exist constants $C_A \in \mathbb{N}$ and $C_H \in \mathbb{N}$ so that for all $s \in \mathcal{R}^{\pm 1}$, $h \in \mathbb{Z}$ and $k \in \{1, \ldots, r\}$ there exists a null $\mathcal{P}$-sequence Σ for the word $\Phi_k(s,h)$ with $\mathrm{Area}(\Sigma) \le C_A$ and $\mathrm{height}_i(\Sigma) \le C_H$ for all i.*

Proof. If $\mathcal{R}$ is empty then there is nothing to prove, so we assume that this is not the case. Note that, by Proposition 11.13 (4), $\Phi_k(s^{-1}, h) \equiv \Phi_k(s, h-\theta_k(s))^{-1} \equiv \Phi_k(s,h)^{-1}$. Thus, by Lemma 11.11, it suffices to consider only those $s \in \mathcal{R}$.

For each $s \in \mathcal{R}$ and $k = 1, \ldots, r$, choose a null $\mathcal{P}$-sequence $\Sigma_{s,k}$ for $\Phi_k(s, 0)$. Define $C_A = \max\{\mathrm{Area}(\Sigma_{s,k}) : s \in \mathcal{R}, 1 \le k \le r\}$ and $C_H = \max\{\mathrm{height}_i(\Sigma_{s,k}) : s \in \mathcal{R}, 1 \le i, k \le r\}$.

Note that, if $i \ne k$, then $\mathrm{height}_i(\Phi_k(s,h)) = \mathrm{height}_i(\Phi_k(s,0)) \le C_H$. Furthermore

$$\mathrm{height}_k(\Phi_k(s,h)) = \begin{cases} 0 & \text{if } h = 0 \text{ and } \mathrm{height}_k(s) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and so $\mathrm{height}_k(\Phi_k(s,h)) \le 1$. Note that $C_H \ge 1$ and so we have that $\mathrm{height}_i(\Phi_k(s,h)) \le C_H$ for all $s \in \mathcal{R}$, $h \in \mathbb{Z}$ and $i, k \in \{1, \ldots, r\}$.

If $s \in \mathcal{R}_2 \cup \ldots \cup \mathcal{R}_n$, then for all $h \in \mathbb{Z}$ we have $\Phi_k(s,h) \equiv \Phi_k(s,0)$ and so the result is immediate. If $s \in \mathcal{R}_1$, then note that $\Phi_k(s,h)$ is freely equal to $(e_k {f_k}^{-1})^h \Phi_k(s,0)(e_k {f_k}^{-1})^{-h}$. The following table presents a null $\mathcal{P}$-scheme for $\Phi_k(s,h)$. In lines 1 and 3 we have used Lemma 11.10.

j	σ_i	Area	height_i
1	$\Phi_k(s,h)$	0	C_H
2	$(e_k {f_k}^{-1})^h \Phi_k(s,0)(e_k {f_k}^{-1})^{-h}$	C_A	C_H
3	$(e_k {f_k}^{-1})^h (e_k {f_k}^{-1})^{-h}$	0	C_H

□

Lemma 11.19. *Suppose $n \geq 3$. Let $s \in \mathcal{C}^{\pm 1}$, $h \in \mathbb{Z}$ and $k \in \{1, \ldots, r\}$. Then there exists a null $\mathcal{P}$-sequence for the word $\Phi_k(s, h)$ with* $\mathrm{Area}(\Sigma) \leq 7(|h| + 1)^2$ *and* $\mathrm{height}_i(\Sigma) \leq 2$ *for each i.*

Proof. By Lemma 11.11 and Proposition 11.13 (4), we may assume that $s \in \mathcal{C}$. We consider 6 disjoint cases. For each case we give a table presenting a null $\mathcal{P}$-sequence for the word $\Phi_k(s, h)$. Say $s \equiv [x, y]$ where $x \in \mathcal{X}_i$ and $y \in \mathcal{X}_j$ and $1 \leq i < j \leq n$.

Case 1. $i, j \geq 2$.

j	σ_j	Area	height_i
1	$x{e_k}^{-\theta_k(x)}y{e_k}^{-\theta_k(y)}{e_k}^{\theta_k(x)}x^{-1}{e_k}^{\theta_k(y)}y^{-1}$	1	2
2	$x{e_k}^{-\theta_k(x)}{e_k}^{\theta_k(x)}y{e_k}^{-\theta_k(y)}x^{-1}{e_k}^{\theta_k(y)}y^{-1}$	0	2
3	$xy{e_k}^{-\theta_k(y)}x^{-1}{e_k}^{\theta_k(y)}y^{-1}$	1	2
4	$xy{e_k}^{-\theta_k(y)}{e_k}^{\theta_k(y)}x^{-1}y^{-1}$	0	2
5	$xyx^{-1}y^{-1}$	1	2
	Total	3	2

Case 2. $i = 1, 2 \leq j \leq n$. $\theta_k(x) = 1$.

j	σ_j	Area	height_i
1	$(e_k{f_k}^{-1})^h e_k{f_k}^{-1}(e_k{f_k}^{-1})^{-h-1}y{e_k}^{-\theta_k(y)}\ldots$ $\ldots(e_k{f_k}^{-1})^{h+1+\theta_k(y)}f_k{e_k}^{-1}(e_k{f_k}^{-1})^{-h-\theta_k(y)}{e_k}^{\theta_k(y)}y^{-1}$	0	1
	Total	0	1

Case 3. $i = 1, 3 \leq j \leq n$. $\theta_k(x) = 0, \theta_k(y) = 0$.

j	σ_j	Area	height_i $(i \neq k)$	height_k
1	$(e_k{f_k}^{-1})^h x(e_k{f_k}^{-1})^{-h}y(e_k{f_k}^{-1})^h x^{-1}(e_k{f_k}^{-1})^{-h}y^{-1}$	$2\|h\|$	2	1
2	$(e_k{f_k}^{-1})^h xy(e_k{f_k}^{-1})^{-h}(e_k{f_k}^{-1})^h x^{-1}(e_k{f_k}^{-1})^{-h}y^{-1}$	0	2	1
3	$(e_k{f_k}^{-1})^h xyx^{-1}(e_k{f_k}^{-1})^{-h}y^{-1}$	$2\|h\|$	2	1
4	$(e_k{f_k}^{-1})^h xyx^{-1}y^{-1}(e_k{f_k}^{-1})^{-h}$	1	2	1
5	$(e_k{f_k}^{-1})^h(e_k{f_k}^{-1})^{-h}$	0	0	1
	Total	$4\|h\| + 1$	2	1

Case 4. $i = 1, 3 \leq j \leq n$. $\theta_k(x) = 0, \theta_k(y) = 1$.

j	σ_j	Area	height_i $(i \neq k)$	height_k
1	$(e_k f_k{}^{-1})^h x (e_k f_k{}^{-1})^{-h} y e_k{}^{-1} (e_k f_k{}^{-1})^{h+1} x^{-1} (e_k f_k{}^{-1})^{-h-1} e_k y^{-1}$	$3\lvert h\rvert$	1	2
2	$(e_k f_k{}^{-1})^h x y e_k{}^{-1} (e_k f_k{}^{-1})^{-h} (e_k f_k{}^{-1})^{h+1} x^{-1} (e_k f_k{}^{-1})^{-h-1} e_k y^{-1}$	0	1	1
3	$(e_k f_k{}^{-1})^h x y e_k{}^{-1} e_k f_k{}^{-1} x^{-1} (e_k f_k{}^{-1})^{-h-1} e_k y^{-1}$	$3\lvert h\rvert$	1	2
4	$(e_k f_k{}^{-1})^h x y e_k{}^{-1} e_k f_k{}^{-1} x^{-1} (e_k f_k{}^{-1})^{-1} e_k y^{-1} (e_k f_k{}^{-1})^{-h}$	0	1	1
5	$(e_k f_k{}^{-1})^h x y f_k{}^{-1} x^{-1} f_k y^{-1} (e_k f_k{}^{-1})^{-h}$	2	1	1
6	$(e_k f_k{}^{-1})^h (e_k f_k{}^{-1})^{-h}$	0	0	1
	Total	$6\lvert h\rvert + 2$	1	2

Case 5. $i = 1, j = 2$. $\theta_k(x) = 0, \theta_k(y) = 0$.

As shorthand, write g_k for the letter $a_k^{(3)} \in \mathcal{X}_3$.

j	σ_j	Area	height_i $(i \neq k)$	height_k
1	$(e_k f_k{}^{-1})^h x (e_k f_k{}^{-1})^{-h} y (e_k f_k{}^{-1})^h x^{-1} (e_k f_k{}^{-1})^{-h} y^{-1}$	0	2	1
2	$(e_k f_k{}^{-1})^h x (e_k f_k{}^{-1})^{-h} (g_k e_k{}^{-1})^{-h} (g_k e_k{}^{-1})^h y \ldots$			
	$\ldots (e_k f_k{}^{-1})^h x^{-1} (e_k f_k{}^{-1})^{-h} y^{-1}$	$2\lvert h\rvert$	2	1
3	$(e_k f_k{}^{-1})^h x (e_k f_k{}^{-1})^{-h} (g_k e_k{}^{-1})^{-h} y \ldots$			
	$\ldots (g_k e_k{}^{-1})^h (e_k f_k{}^{-1})^h x^{-1} (e_k f_k{}^{-1})^{-h} y^{-1}$	$\frac{3}{2}\lvert h\rvert(\lvert h\rvert + 1)$	2	2
4	$(e_k f_k{}^{-1})^h x (g_k f_k{}^{-1})^{-h} y \ldots$			
	$\ldots (g_k e_k{}^{-1})^h (e_k f_k{}^{-1})^h x^{-1} (e_k f_k{}^{-1})^{-h} y^{-1}$	$\frac{3}{2}\lvert h\rvert(\lvert h\rvert + 1)$	2	2
5	$(e_k f_k{}^{-1})^h x (g_k f_k{}^{-1})^{-h} y (g_k f_k{}^{-1})^h x^{-1} (e_k f_k{}^{-1})^{-h} y^{-1}$	$2\lvert h\rvert$	2	1
6	$(e_k f_k{}^{-1})^h (g_k f_k{}^{-1})^{-h} x y (g_k f_k{}^{-1})^h x^{-1} (e_k f_k{}^{-1})^{-h} y^{-1}$	$2\lvert h\rvert$	2	1
7	$(e_k f_k{}^{-1})^h (g_k f_k{}^{-1})^{-h} x y x^{-1} (g_k f_k{}^{-1})^h (e_k f_k{}^{-1})^{-h} y^{-1}$	$\frac{3}{2}\lvert h\rvert(\lvert h\rvert + 1)\rvert$	2	2
8	$(e_k g_k{}^{-1})^h x y x^{-1} (g_k f_k{}^{-1})^h (e_k f_k{}^{-1})^{-h} y^{-1}$	$\frac{3}{2}\lvert h\rvert(\lvert h\rvert + 1)\rvert$	2	2
9	$(e_k g_k{}^{-1})^h x y x^{-1} (g_k e_k{}^{-1})^h y^{-1}$	$2\lvert h\rvert$	2	1
10	$(e_k g_k{}^{-1})^h x y x^{-1} y^{-1} (g_k e_k{}^{-1})^h$	1	2	1
11	$(e_k g_k{}^{-1})^h (g_k e_k{}^{-1})^h$	0	0	1
	Total	$6\lvert h\rvert^2 + 14\lvert h\rvert + 1$	2	2

Case 6. $i = 1, j = 2$. $\theta_k(x) = 0, \theta_k(y) = 1$.

j	σ_j	Area	height_i $(i \neq k)$	height_k
1	$(e_k f_k{}^{-1})^h x (e_k f_k{}^{-1})^{-h} f_k e_k{}^{-1} \ldots$			
	$\ldots (e_k f_k{}^{-1})^{h+1} x^{-1} (e_k f_k{}^{-1})^{-h-1} e_k f_k{}^{-1}$	0	1	1
	Total	0	1	1

$\square$

Proposition 11.20. *Suppose $n \geq 3$. Then there exist constants $C'_A \in \mathbb{N}$ and $C'_H \in \mathbb{N}$ so that for any $k \in \{1, \ldots, r\}$ and any $\mathcal{P}$-expression $\mathcal{E}$ for a word*

$w \in \mathcal{X}^{\pm *}$ *there exists a $\mathcal{P}$-expression $\overline{\mathcal{E}}$ for w with*

$$\operatorname{Area}(\overline{\mathcal{E}}) \leq C'_A \operatorname{Area}(\mathcal{E})(\operatorname{height}_k(\mathcal{E})+1)^2 + 2|w|(\operatorname{height}_k(w)+1)^2$$

$$\operatorname{height}_i(\overline{\mathcal{E}}) \leq \begin{cases} \max\{\operatorname{height}_i(w)+1, C'_H\} & \text{if } i = k, \\ \max\{\operatorname{height}_i(w)+1, C'_H, \operatorname{height}_i(\mathcal{E})\} & \text{if } i \neq k. \end{cases}$$

Proof. Say $\mathcal{E} = (x_j, r_j)_{j=1}^m$. Let C_A and C_H be the constants given by Lemma 11.18 and define $C'_A = \max\{C_A, 7\}$ and $C'_H = \max\{C_H, 2\}$. Then, by Lemmas 11.18, 11.19 and 11.9, for each j there exists a $\mathcal{P}$-expression $\mathcal{E}_j$ for $\Phi_k(r_j, \theta_k(x_j))$ with $\operatorname{Area}(\mathcal{E}_j) \leq C'_A(|\theta_k(x_j)|+1)^2 \leq C'_A(\operatorname{height}_k(\mathcal{E})+1)^2$ and $\operatorname{height}_i(\mathcal{E}_j) \leq C'_H$ for all i. Say $\mathcal{E}_j = (x_{jl}, r_{jl})_{l=1}^{m_j}$. For each j, define $\mathcal{E}'_j$ to be the $\mathcal{P}$-expression $(\Phi_k(x_j, 0)x_{jl}, r_{jl})_{l=1}^{m_j}$ and define $\mathcal{E}'$ to be the $\mathcal{P}$-expression $\mathcal{E}'_1 \ldots \mathcal{E}'_m$. Then $\operatorname{Area}(\mathcal{E}') = \sum_{j=1}^m m_j \leq C'_A \operatorname{Area}(\mathcal{E})(\operatorname{height}_k(\mathcal{E})+1)^2$ and

$$\begin{aligned}
\operatorname{height}_i(\mathcal{E}') &\leq \max_{\substack{1 \leq j \leq m \\ 1 \leq l \leq m_j}} \{\operatorname{height}_i(\Phi_k(x_j, 0)x_{jl})\} \\
&\leq \max_{\substack{1 \leq j \leq m \\ 1 \leq l \leq m_j}} \{\operatorname{height}_i(\Phi_k(x_j, 0)), \operatorname{height}_i(x_{jl})\} \\
&\leq \max_{1 \leq j \leq m} \{\operatorname{height}_i(\Phi_k(x_j, 0)), C'_H\} \\
&\leq \begin{cases} \max\{1, C'_H\} & i = k, \\ \max\{\operatorname{height}_i(x_j), C'_H\} & i \neq k \end{cases} \\
&\leq \begin{cases} C'_H & i = k, \\ \max\{\operatorname{height}_i(\mathcal{E}), C'_H\} & i \neq k, \end{cases}
\end{aligned}$$

where we have made use of Proposition 11.13 (3) and the fact that $\theta_i(\Phi_k(x_j, 0)) = 0$.

Furthermore

$$\begin{aligned}
\partial\mathcal{E}' &\equiv \prod_{j=1}^{m} \partial\mathcal{E}'_j \\
&\equiv \prod_{j=1}^{m}\prod_{l=1}^{m_j} \Phi_k(x_j,0)x_{jl}r_{jl}{x_{jl}}^{-1}\Phi_k(x_j,0)^{-1} \\
&\stackrel{\text{fr}}{=} \prod_{j=1}^{m} \Phi_k(x_j,0)\left(\prod_{l=1}^{m_j} x_{jl}r_{jl}{x_{jl}}^{-1}\right)\Phi_k(x_j,0)^{-1} \\
&\equiv \prod_{j=1}^{m} \Phi_k(x_j,0)\partial\mathcal{E}_j\Phi_k(x_j,0)^{-1} \\
&\stackrel{\text{fr}}{=} \prod_{j=1}^{m} \Phi_k(x_j,0)\Phi_k(r_j,\theta_k(x_j))\Phi_k(x_j,0)^{-1} \\
&\equiv \prod_{j=1}^{m} \Phi_k(x_jr_j,0)\Phi_k({x_j}^{-1},\theta_k(x_j)) \\
&\equiv \prod_{j=1}^{m} \Phi_k(x_jr_j,0)\Phi_k({x_j}^{-1},\theta_k(x_jr_j)) \\
&\equiv \prod_{j=1}^{m} \Phi_k(x_jr_j{x_j}^{-1},0) \\
&\equiv \Phi_k\left(\prod_{j=1}^{m} x_jr_j{x_j}^{-1},0\right) \\
&\equiv \Phi_k\left(\partial\mathcal{E},0\right) \\
&\stackrel{\text{fr}}{=} \Phi_k(w,0),
\end{aligned}$$

where we have made use of Proposition 11.13 (4), (5) and (6).

Since w is null-homotopic, $\theta_k(w)=0$ and hence, by Proposition 11.13 (1), $\Phi_k(w,0)=w$ in D. By Lemma 11.17, there exists a $\mathcal{P}$-sequence $\Sigma=(\sigma_j)_{j=0}^m$ converting $\Phi_k(w,0)$ to w with $\text{Area}(\Sigma)\le 2|w|(\text{height}_k(w)+1)^2$ and $\text{height}_i(\Sigma)\le \text{height}_i(w)+1$ for each i. Let Σ^{-1} be the $\mathcal{P}$-sequence $\sigma_m,\sigma_{m-1},\dots,\sigma_0$ converting w to $\Phi_k(w,0)$. Then $\text{Area}(\Sigma^{-1})=\text{Area}(\Sigma)$ and $\text{height}_i(\Sigma^{-1})=\text{height}_i(\Sigma)$ for each i. By Lemma 11.9, there exists a $\mathcal{P}$-expression $\mathcal{E}''$ for $w\left(\Phi_k(w,0)\right)^{-1}$ with $\text{Area}(\mathcal{E}'')=\text{Area}(\Sigma^{-1})$ and $\text{height}_i(\mathcal{E}'')\le\text{height}_i(\Sigma^{-1})$ for each i. Define $\overline{\mathcal{E}}=\mathcal{E}''\mathcal{E}'$. Then $\overline{\mathcal{E}}$ is a $\mathcal{P}$-expression for w with the required bounds on its area and heights. ☐

Proposition 11.21. *Suppose $n\ge 3$. Then there exist constants $C''_A\in\mathbb{N}$ and $C''_H\in\mathbb{N}$ so that given any $\mathcal{P}$-expression $\mathcal{E}$ for a word $w\in\mathcal{X}^{\pm *}$ there exists a*

$\mathcal{P}$-expression $\overline{\mathcal{E}}$ for w with

$$\operatorname{Area}(\overline{\mathcal{E}}) \leq C''_A(\operatorname{Area}(\mathcal{E}) + |w|) \prod_{j=1}^{r} \zeta_j^2,$$
$$\operatorname{height}_i(\overline{\mathcal{E}}) \leq \max\{\operatorname{height}_i(w) + 1, C''_H\},$$

where $\zeta_j = \max\{\operatorname{height}_j(w) + 1, \operatorname{height}_j(\mathcal{E}) + 1, C''_H\}$.

Proof. Let C'_A and C'_H be the constants given by Proposition 11.20. We claim that, for each $l \in \{0, 1, \ldots, r\}$, there exists a $\mathcal{P}$-expression $\mathcal{E}_l$ for w with

$$\operatorname{Area}(\mathcal{E}_l) \leq (C'_A)^{l-1}(C'_A \operatorname{Area}(\mathcal{E}) + 2l|w|) \prod_{j=1}^{l} \zeta_j^2$$
$$\operatorname{height}_i(\mathcal{E}_l) \leq \begin{cases} \max\{\operatorname{height}_i(w) + 1, C'_H\} & \text{if } 1 \leq i \leq l, \\ \max\{\operatorname{height}_i(w) + 1, C'_H, \operatorname{height}_i(\mathcal{E}) + 1\} & \text{if } l + 1 \leq i \leq r. \end{cases}$$

The claim is proved by induction on l. Set $\mathcal{E}_0 = \mathcal{E}$ and, given that $\mathcal{E}_{l-1}$ has been defined, define $\mathcal{E}_l$ be the $\mathcal{P}$-expression given by applying Proposition 11.20 to $\mathcal{E}_{l-1}$ with $k = l$. Then $\mathcal{E}_l$ certainly satisfies the required bounds on its heights and

$$\begin{aligned}
\operatorname{Area}(\mathcal{E}_l) &\leq C'_A \operatorname{Area}(\mathcal{E}_{l-1})(\operatorname{height}_l(\mathcal{E}_{l-1}) + 1)^2 + 2|w|(\operatorname{height}_l(w) + 1)^2 \\
&\leq C'_A \operatorname{Area}(\mathcal{E}_{l-1})\zeta_l^2 + 2|w|\zeta_l^2 \\
&\leq (C'_A)^{l-1}(C'_A \operatorname{Area}(\mathcal{E}) + 2(l-1)|w|) \prod_{j=1}^{l} \zeta_j^2 + 2|w|\zeta_l^2 \\
&\leq (C'_A)^{l-1}(C'_A \operatorname{Area}(\mathcal{E}) + 2(l-1)|w|) \prod_{j=1}^{l} \zeta_j^2 + 2(C'_A)^l|w| \prod_{j=1}^{l} \zeta_j^2 \\
&\leq (C'_A)^{l-1}(C'_A \operatorname{Area}(\mathcal{E}) + 2l|w|) \prod_{j=1}^{l} \zeta_j^2
\end{aligned}$$

as required.

The proposition now follows by setting $C''_A = (C'_A)^{l-1} \max\{C'_A, 2r\}$ and $C''_H = C'_H$ and taking $\overline{\mathcal{E}}$ to be $\mathcal{E}_r$. □

Proposition 11.22. *For each $i = 1, \ldots, n$, let (α_i, ρ_i) be an area-radius pair for some finite presentation of Γ_i, and define $\alpha(l) = \max(\{l^2\} \cup \{\alpha_i(l) : 1 \leq i \leq n\})$ and $\rho(l) = \max(\{l\} \cup \{\rho_i(l) : 1 \leq i \leq n\})$. Then there exist functions $\overline{\alpha}, \overline{\rho} : \mathbb{N} \to \mathbb{N}$ with $\overline{\alpha} \simeq \alpha$ and $\overline{\rho} \simeq \rho$ such that the following property holds: For any null-homotopic word $w \in \mathcal{X}^{\pm *}$, there exists a $\mathcal{P}$-expression $\mathcal{E}$ for w with $\operatorname{Area}(\mathcal{E}) \leq \overline{\alpha}(|w|)$ and $\operatorname{height}_j(\mathcal{E}) \leq \overline{\rho}(|w|)$ for each j.*

Proof. Proposition 3.15 shows that, for each $i = 1, \ldots, n$, there exist functions $\alpha'_i, \rho'_i : \mathbb{N} \to \mathbb{N}$ with $\alpha'_i \simeq \alpha_i$ and $\rho'_i \simeq \rho_i$ so that (α'_i, ρ'_i) is an area-radius pair for $\mathcal{P}_i$. Define functions $\alpha', \rho' : \mathbb{N} \to \mathbb{N}$ by $\alpha'(l) = \max(\{l^2\} \cup \{\alpha'_i(l) : 1 \leq i \leq n\})$ and $\rho'(l) = \max(\{l\} \cup \{\rho'_i(l) : 1 \leq i \leq n\})$. By the same reasoning as in Proposition 11.4, one sees that $\alpha' \simeq \alpha$ and $\rho' \simeq \rho$.

Now let $w \in \mathcal{X}^{\pm *}$ be a null-homotopic word. For each $i = 1, \dots n$, define w_i to be the word $p_i(w)$, where p_i is the projection map $\mathcal{X}^{\pm *} \to \mathcal{X}_i^{\pm *}$. Then $w = w_1 \dots w_n$ in D, each w_i is null-homotopic and $|w| = |w_1 \dots w_n|$. Observe that there exists a $\mathcal{P}$-sequence $\Sigma = (\sigma_l)_{l=1}^m$ converting w to $w_1 \dots w_n$ with area at most $|w|^2$ and with each σ_{i+1} being obtained from σ_i by applying a relator from $\mathcal{C}$. It follows that, for each l, $\mathrm{height}_j(\sigma_l) \leq |\sigma_l| = |w|$. Thus, by Lemma 11.9, there exists a $\mathcal{P}$-expression $\mathcal{E}'$ for $w(w_1 \dots w_n)^{-1}$ with $\mathrm{Area}(\mathcal{E}') \leq |w|^2$ and $\mathrm{height}_j(\mathcal{E}') \leq |w|$.

For each i, let $\mathcal{E}_i$ be a $\mathcal{P}_i$-expression for w_i with $\mathrm{Area}(\mathcal{E}_i) \leq \alpha_i'(|w_i|) \leq \alpha_i'(|w|) \leq \alpha'(|w|)$ and $\mathrm{Rad}(\mathcal{E}_i) \leq \rho_i'(|w_i|) \leq \rho_i'(|w|) \leq \rho'(|w|)$. Then $\mathrm{height}_j(\mathcal{E}_i) \leq \mathrm{Rad}(\mathcal{E}_i) \leq \rho'(|w|)$. If we set $\mathcal{E}''$ to be the $\mathcal{P}$-expression $\mathcal{E}_1 \dots \mathcal{E}_n$ then $\mathrm{Area}(\mathcal{E}'') \leq n\alpha'(|w|)$ and $\mathrm{height}_j(\mathcal{E}'') \leq \rho'(|w|)$. Define $\mathcal{E} = \mathcal{E}'\mathcal{E}''$. Then $\mathrm{Area}(\mathcal{E}) \leq |w|^2 + n\alpha'(|w|) \leq (n+1)\alpha'(|w|)$ and $\mathrm{height}_j(\mathcal{E}) \leq \max\{|w|, \rho'(|w|)\} \leq \rho'(|w|)$. Define $\overline{\alpha}$ and $\overline{\rho}$ by $\overline{\alpha}(l) = (n+1)\alpha'(l)$ and $\overline{\rho}(l) = \rho'(l)$. □

Proof of Theorem 11.15. Suppose that $w \in \mathcal{X}^{\pm *}$ is a null-homotopic word with $w \not\equiv \emptyset$, and let C_A'' and C_H'' be the constants given by Proposition 11.21. Since w is null-homotopic, there exists a $\mathcal{P}$-expression $\mathcal{E}$ for w, and so Proposition 11.21 implies that there exists a $\mathcal{P}$-expression $\overline{\mathcal{E}}$ for w with $\mathrm{height}_i(\overline{\mathcal{E}}) \leq \max\{\mathrm{height}_i(w)+1, C_H''\}$ for each i. Lemma 11.8 therefore gives that $\mathrm{Dep}_{\mathcal{X}}(\overline{\mathcal{E}}, \ker\theta) \leq r\max\{\mathrm{Dep}_{\mathcal{X}}(w, \ker\theta) + 1, C_H''\}$. By Theorem 11.12, $\ker\theta$ is finitely generated and so Proposition 10.4 (1) implies that $\ker\theta$ is finitely presented.

Now suppose that, for each i, (α_i, ρ_i) is an area-radius pair for some finite presentation of Γ_i. Let $\overline{\alpha}$ and $\overline{\rho}$ be as given by Proposition 11.22. Then we can take $\mathcal{E}$ to have $\mathrm{Area}(\mathcal{E}) \leq \overline{\alpha}(|w|)$ and $\mathrm{height}_i(\mathcal{E}) \leq \overline{\rho}(|w|)$ for each i. Thus, by Proposition 11.21,

$$\begin{aligned}
\mathrm{Area}(\overline{\mathcal{E}}) &\leq C_A''(\overline{\alpha}(|w|) + |w|)\prod_{i=1}^{r}\left(\max\{\mathrm{height}_i(w) + 1, \overline{\rho}(|w|) + 1, C_H''\}\right)^2 \\
&\leq 2C_A''\overline{\alpha}(|w|)\left(\max\{|w| + 1, \overline{\rho}(|w|) + 1, C_H''\}\right)^{2r} \\
&\leq 2C_A''\overline{\alpha}(|w|)\left(\max\{\overline{\rho}(|w|) + 1, C_H''\}\right)^{2r} \\
&\leq 2C_A''\overline{\alpha}(|w|)\left(2C_H''\overline{\rho}(|w|)\right)^{2r} \\
&\leq 2^{2r+1}C_A''(C_H'')^{2r}\overline{\alpha}(|w|)\overline{\rho}(|w|)^{2r}.
\end{aligned}$$

Therefore, by Proposition 10.4 (2), $\overline{\alpha}\,\overline{\rho}^{2r}$ is an isoperimetric function for H. Since $\alpha(l), \rho(l), \overline{\alpha}(l), \overline{\rho}(l)$ are all $\geq l$, it follows that $\overline{\alpha}\,\overline{\rho}^{2r} \simeq \alpha\rho^{2r}$ and so $\alpha\rho^{2r}$ is an isoperimetric function for H. □

11.7 Isoperimetric functions 2

In this section we prove Theorem 11.3 (4), which will follow directly from Corollary 11.24 and Proposition 11.4. The following result generalises [14, Theorem 2.1] which treats the $n = 3, r = 1$ case.

Theorem 11.23. *Let θ be a homomorphism from a direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$ of $n \geq 3$ finitely presented groups to a finitely generated free abelian group A such that the restriction of θ to each factor Γ_i is surjective. Suppose that $n \geq 2r$, where $r = \dim A \otimes \mathbb{Q}$. Then $\ker\theta$ is finitely presented.*

Define $D_1 = \Gamma_1 \times \ldots \times \Gamma_{n-r}$ and $D_2 = \Gamma_{n-r+1} \times \ldots \times \Gamma_n$. Let Δ be the distortion function of $\ker\theta \cap D_1$ *in D_1 with respect to some choice of finite generating sets and let β_1 and β_2 be the Dehn functions of D_1 and D_2 respectively with respect to some choice of finite presentations. Then there exist functions $\Delta' \simeq \Delta$, $\beta_1' \simeq \beta_1$ and $\beta_2' \simeq \beta_2$ so that the function β' defined by*

$$\beta'(l) = l\beta_1'(\Delta'(l)) + \beta_2'(l)$$

is an isoperimetric function for $\ker\theta$. *Furthermore, Δ', β_1' and β_2' can be chosen to be increasing and superlinear.*

Here a function $f : \mathbb{N} \to \mathbb{N}$ is said to be *superlinear* if $f(l) \geq l$. Note that the conditions $n \geq 2r$ and $n \geq 3$ imply that $n - r \geq 2$. Thus $\ker\theta \cap D_1$ is the fibre product of the homomorphisms $\theta|_{\Gamma_1}$ and $-\theta|_{\Gamma_2 \times \ldots \times \Gamma_{n-r}}$ and hence is finitely generated by Lemma 9.5. The function Δ is therefore well-defined up to $\approx$-equivalence and hence, in particular, up to the the weaker $\simeq$-equivalence.

Proof. Let $x_1, \ldots, x_r$ be a free abelian basis for A. Note that the condition $n \geq 2r$ implies that $n - r \geq r$. For each $i = 1, \ldots, r$, let $t_i \in \Gamma_i$ and $t_i' \in \Gamma_{n-r+i}$ be such that $\theta(t_i) = \theta(t_i') = x_i$. Define $\mathcal{T} = \{t_1, \ldots, t_r\}$ and $\mathcal{T}' = \{t_1', \ldots, t_r'\}$. Define $\mathcal{A} = \{a_1, \ldots, a_r\}$, where $a_i = t_i(t_i')^{-1}$.

Let $\mathcal{B}_1$ be a finite generating set for $K = \ker\theta \cap D_1$. For each $t \in \mathcal{T}$, $b \in \mathcal{B}_1$ and $\epsilon \in \{\pm 1\}$, let $w_{bt\epsilon} \in \mathcal{B}_1^{\pm *}$ be a word representing $t^\epsilon b t^{-\epsilon}$. Let $\mathcal{P}_1 = \langle \mathcal{B}_1, \mathcal{T} \,|\, \mathcal{S} \rangle$ be a finite presentation for D_1 where $\mathcal{S}$ includes all relations $t^\epsilon b t^{-\epsilon} w_{bt\epsilon}^{-1}$. Let $\bar{\beta}_1$ be the Dehn function of $\mathcal{P}_1$ and define $\beta_1'(l) = \bar{\beta}_1(l) + l$. Then β_1' is increasing and superlinear and $\beta_1' \simeq \beta_1$.

Let $\mathcal{B}_2 \subseteq \ker\theta \cap D_2$ be a finite collection of elements such that $\mathcal{B}_2 \cup \mathcal{T}'$ generates D_2. Note that $\mathcal{B}_2$ may not generate $ker\theta \cap D_2$, which may not even be finitely generated. Define a homomorphism $s : D_2 \to \ker\theta$ by mapping each word $w(\mathcal{B}_2, \mathcal{T}') \mapsto w(\mathcal{B}_2, \mathcal{A}^{-1})$. Note that s does indeed define a homomorphism since if $w(\mathcal{B}_1, \mathcal{T}')$ is null-homotopic then its exponent sum in each letter of $\mathcal{T}'$ is 0 and hence $w(\mathcal{B}_1, \mathcal{T}')$ and $w(\mathcal{B}_1, \mathcal{A}^{-1})$ define the same group element. Observe that s is a splitting of the short exact sequence $1 \to K \to \ker\theta \to D_2 \to 1$ where the homomorphism $\ker\theta \to D_2$ is the projection homomorphism. Define $H \cong D_2$ to be the image of s. Then $\ker\theta \cong K \rtimes H$. Let $\mathcal{P}_2 = \langle \mathcal{A}, \mathcal{B}_2 \,|\, \mathcal{R} \rangle$ be a finite presentation for H and let $\bar{\beta}_2$ be the Dehn function of $\mathcal{P}_2$. Define $\beta_2'(l) = \bar{\beta}_2(l) + l$. Then β_2' is increasing and superlinear and $\beta_2' \simeq \beta_2$.

Define $\mathcal{S}' = \{s(\mathcal{A}, \mathcal{B}_1) : s(\mathcal{T}, \mathcal{B}_1) \in \mathcal{S}\}$ and note that the words in $\mathcal{S}'$ are null-homotopic. Define $\mathcal{C} = \{[b_1, b_2] : b_i \in \mathcal{B}_i\}$.

Claim. $\ker\theta$ *is presented by $\mathcal{P} = \langle \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \,|\, \mathcal{R}, \mathcal{S}', \mathcal{C} \rangle$.*

To prove the claim, suppose that $w = w(\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$ is a null-homotopic word. By applying relations from $\mathcal{C}$ and $\mathcal{S}'$ we can convert w to a word $w_1 w_2$, where $w_1 = w_1(\mathcal{B}_1)$ and $w_2 = w_2(\mathcal{A}, \mathcal{B}_2)$. Then w_2 is null-homotopic in H and so $w_1 w_2$ can be converted to w_1 by applying relators from $\mathcal{R}$. Furthermore, w_1 is null-homotopic in D_1, so there exists a free equality $w_1(\mathcal{B}_1) \overset{\text{fr}}{=} \prod u_i(\mathcal{T}, \mathcal{B}_1) s_i(\mathcal{T}, \mathcal{B}_1) u_i^{-1}(\mathcal{T}, \mathcal{B}_1)$ for some words u_i and some relators $s_i \in \mathcal{S}$. Thus $w_1(\mathcal{B}_1) \overset{\text{fr}}{=} u_i(\mathcal{A}, \mathcal{B}_1) s_i(\mathcal{A}, \mathcal{B}_1) u_i^{-1}(\mathcal{A}, \mathcal{B}_1)$, completing the proof of the claim.

A priori, the above scheme gives an exponential isoperimetric function for $\ker\theta$. We now show how this can be improved. Let $\bar{\Delta}$ be the distortion function

of K in D_1 with respect to the generating sets $\mathcal{B}_1$ and $\mathcal{B}_1 \cup \mathcal{T}$. Define $\Delta'(l) = \bar{\Delta}(l) + l$. Then Δ' is increasing and superadditive. Furthermore, $\Delta' \simeq \Delta$ since $\bar{\Delta} \approx \Delta$.

Claim. *Let $\sigma = \sigma(\mathcal{A}, \mathcal{B}_1)$ be a word of length at most l having exponent sum 0 in each letter $a \in \mathcal{A}$. Let $b \in \mathcal{B}_2$. Then $\mathrm{Area}_{\mathcal{P}}([b, \sigma]) \leq 3\beta_1'(\Delta'(l))$.*

To prove the claim, note that, since $\sigma(\mathcal{A}, \mathcal{B}_1)$ has exponent sum 0 in each $a \in \mathcal{A}$, it represents the same element of K as $\sigma(\mathcal{T}, \mathcal{B}_1)$. It is thus represented by some word $\tau = \tau(\mathcal{B}_1)$ with $|\tau| \leq \overline{\Delta}(l)$. Then $\sigma(\mathcal{T}, \mathcal{B}_1)\tau^{-1}(\mathcal{B}_1)$ is null-homotopic and so there exists a null $\mathcal{P}_1$-expression $(\rho_i(\mathcal{T}, \mathcal{B}_1), s_i(\mathcal{T}, \mathcal{B}_1))$ for $\sigma(\mathcal{T}, \mathcal{B}_1)\tau^{-1}(\mathcal{B}_1)$ with area at most $\beta_1'(\overline{\Delta}(l) + l)$. Thus $(\rho_i(\mathcal{A}, \mathcal{B}_1), s_i(\mathcal{A}, \mathcal{B}_1))$ is a null $\mathcal{P}$-expression for $\sigma(\mathcal{A}, \mathcal{B}_1)\tau^{-1}(\mathcal{B}_1)$ and so there exists a $\mathcal{P}$-sequence converting $b\sigma(\mathcal{A}, \mathcal{B}_1)$ to $b\tau(\mathcal{B}_1)$ with area at most $\beta_1'(\overline{\Delta}(l) + l) \leq \beta_1'(\Delta'(l))$.

By applying relators from $\mathcal{C}$, we see that there exists a $\mathcal{P}$-sequence converting $b\tau(\mathcal{B}_1)$ to $\tau(\mathcal{B}_1)b$ with area at most $|\tau| \leq \overline{\Delta}(l)$. Finally, we can convert $\tau(\mathcal{B}_1)b$ to $\sigma(\mathcal{A}, \mathcal{B}_1)b$ by a $\mathcal{P}$-sequence of area at most $\beta_1'(\Delta'(l))$. Thus $\mathrm{Area}_{\mathcal{P}}([b, \sigma]) \leq 2\beta_1'(\Delta'(l)) + \overline{\Delta}(l) \leq 3\beta_1'(\Delta'(l))$, completing the proof of the claim.

Now, to obtain the stated isoperimetric function for $\ker\theta$, let $w \in (\mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2)^{\pm *}$ be a null-homotopic word in $\ker\theta$. Then w can be written as $u_0x_1u_1x_2 \ldots x_nu_n$ where each $x_i \in (\mathcal{A} \cup \mathcal{B}_2)^{\pm 1}$ and each $u_i \in \mathcal{B}_1^{\pm *}$ is some (possibly empty) word.

We will define a sequence of words $U_0, \ldots, U_n \in (\mathcal{A} \cup \mathcal{B}_1)^{\pm *}$ with each U_i having zero exponent sum in each letter $a \in \mathcal{A}$ and with the word

$$w_i \equiv u_0x_1 \ldots u_{n-i-1}x_{n-i}U_ix_{n-i+1}x_{n-i+2} \ldots x_n$$

representing the same element as w. Take $U_0 \equiv u_n$ and define the subsequent U_i recursively as follows. If $x_{n-i} \in \mathcal{A}$ then define $U_{i+1} :\equiv u_{n-i-1}x_{n-i}U_ix_{n-i}^{-1}$. If $x_{n-i} \in \mathcal{B}_2$ then define $U_{i+1} :\equiv u_{n-i-1}U_i$. In the former case we see that w_{i+1} is freely equal to w_i. In the latter case the above claim shows that there exists a $\mathcal{P}$-sequence converting w_i to w_{i+1} with area at most $3\beta_1'(\Delta'(|U_{i-1}|))$. Now, $|U_i| \leq |u_{n-i}| + |U_{i-1}| + 2 \leq |u_{n-i}| + |u_{n-i+1}| + \ldots + |u_n| + 2i \leq 2|w|$. Thus there exists a $\mathcal{P}$-sequence converting w to $w_n \equiv U_nx_1 \ldots x_n$ with area at most $3n\beta_1'(\Delta'(2|w|)) \leq 3|w|\beta_1'(\Delta'(2|w|))$.

Note that $x_1 \ldots x_n$ represents an element of H and, since $U_n = U_n(\mathcal{A}, \mathcal{B}_1)$ has exponent sum 0 in each letter $a \in \mathcal{A}$, that U_n represents an element of K. Thus, since $U_nx_1 \ldots x_n$ represents the identity in the semidirect product $K \rtimes H$, it follows that U_n and $x_1 \ldots x_n$ are both null-homotopic. Since $U_n = U_n(\mathcal{A}, \mathcal{B}_1)$ has exponent sum 0 in each letter $a \in \mathcal{A}$ it represents the same element as $U_n(\mathcal{T}, \mathcal{B}_1)$. Let $(\rho_i(\mathcal{T}, \mathcal{B}_1), s_i(\mathcal{T}, \mathcal{B}_1))$ be a $\mathcal{P}_1$-expression for $U_n(\mathcal{T}, \mathcal{B}_1)$ with area at most $\beta_1'(|U_n|) \leq \beta_1'(2|w|)$. Then $(\rho_i(\mathcal{A}, \mathcal{B}_1), s_i(\mathcal{A}, \mathcal{B}_1))$ is a $\mathcal{P}$-expression for $U_n(\mathcal{A}, \mathcal{B}_1)$, so $\mathrm{Area}_{\mathcal{P}}(U_n) \leq \beta_1'(2|w|)$. Any $\mathcal{P}_2$-expression for $x_1 \ldots x_n$ is also a $\mathcal{P}$-expression for $x_1 \ldots x_n$, so $\mathrm{Area}_{\mathcal{P}}(x_1 \ldots x_n) \leq \beta_2'(n) \leq \beta_2'(|w|)$. Putting these bounds together demonstrates that $\mathrm{Area}_{\mathcal{P}}(w) \leq 3|w|\beta_1'(\Delta'(2|w|) + \beta_1'(2|w|) + \beta_2'(|w|) \leq 4|w|\beta_1'(\Delta'(2|w|)) + \beta_2'(|w|)$. □

Corollary 11.24. *We continue with the notation and hypotheses of Theorem 11.23. Then the function β defined by*

$$\beta(l) = l\beta_1(l^2) + \beta_2(l)$$

is an isoperimetric function for $\ker\theta$.

Proof. Since A is abelian it admits a quadratic isoperimetric function. Thus, by Lemma 9.5, together with the definition of $\simeq$-equivalence, there exists $K \in \mathbb{N}$ so that the function Δ satisfies $\Delta(l) \leq Kl^2$. By the definition of $\simeq$-equivalence, there exists $C \in \mathbb{N}$ so that $\Delta'(l) \leq Cl^2$. Since β_1' is increasing, it follows that the function $\bar{\beta}$, defined by $\bar{\beta}(l) = l\beta_1'(Cl^2) + \beta_2'(l)$, is an isoperimetric function for $\ker\theta$.

Note that the conditions $n \geq 2r$ and $n \geq 3$ imply that $n - r \geq 2$. If $r = 0$ then the result is trivial, so we may assume $r \geq 1$. Since the restriction of θ to each of the Γ_i is surjective, each Γ_i contains an element of infinite order, and hence D_1 contains $\mathbb{Z}^2$ as a subgroup. By [15, Theorem 6.1.10 (1)] D_1 is thus not hyperbolic and hence by Lemma 11.6 there exists $C' \in \mathbb{N}$ so that $l^2 \leq C'\beta_1(l) + C'$.

Let $M_1 \in \mathbb{N}$ and $M_2 \in \mathbb{N}$ be the constants arising in the definition of β_1' and β_2' being $\preceq \beta_1$ and β_2 respectively. Define $M = \max\{M_1, M_2\}$. Then

$$\begin{aligned}
\bar{\beta}(l) &= l\beta_1'(Cl^2) + \beta_2'(l) \\
&\leq l[M\beta_1(M(Cl^2) + M) + MCl^2 + M] + M\beta_2(Ml + M) + Ml + M \\
&= Ml\beta_1(MCl^2 + M) + M\beta_2(Ml + M) + MCl^3 + 2Ml + M \\
&\leq Ml\beta_1(MCl^2 + M) + M\beta_2(Ml + M) + MCl(C'\beta_1(l) + C') + 2Ml + M \\
&\leq M(CC' + 1)l\beta_1(MCl^2 + M) + M\beta_2(Ml + M) + (CC' + 2)Ml + M \\
&\leq M(CC' + 1)\beta(MCl + M) + (CC' + 2)Ml + M.
\end{aligned}$$

Thus $\bar{\beta} \preceq \beta$ and so β is an isoperimetric function for $\ker\theta$. □

12 Depth of subdirect products

12.1 Definition

Definition 12.1. Let $D = \Gamma_1 \times \ldots \times \Gamma_n$ be a direct product of groups. Write $\mathcal{L}_n$ for the lattice of subsets of $\{1, \ldots, n\}$. Given a subset $\mathcal{S} = \{i_1, \ldots, i_k\} \in \mathcal{L}_n$, define $D_{\mathcal{S}}$ to be the direct product $\Gamma_{i_1} \times \ldots \times \Gamma_{i_k}$ and define $p_{\mathcal{S}}$ to be the projection homomorphism $D \to D_{\mathcal{S}}$.

The *depth* of a subgroup $H \leq D$ is defined to be

$$\mathrm{Depth}(H) = n - \max\{k : [D_{\mathcal{S}} : p_{\mathcal{S}}(H)] < \infty \text{ for all } \mathcal{S} \in \mathcal{L}_n \text{ with } |\mathcal{S}| = k\}.$$

We remark that the depth of a subgroup $H \leq D$ depends on the choice of a particular decomposition of D as a direct product. Also note that if D has n factors then $0 \leq \mathrm{Depth}(H) \leq n$. The depth 0 subgroups are precisely the finite index subgroups of D; the depth 1 subgroups are precisely the virtually-full subgroups of D; and the depth $n-1$ subgroups are precisely the subdirect products of finite index subgroups of D. The following lemma shows that the definition of depth given here agrees with the definition of depth given by Meinert [32] for coabelian subgroups $H \leq D$.

Lemma 12.2. *Let H be a coabelian subgroup of the direct product $D = \Gamma_1 \times \ldots \times \Gamma_n$ with quotient homomorphism $\theta : D \to D/H$. Then*

$$\begin{aligned}
\mathrm{Depth}(H) &= \min\{k : [D : D_{\mathcal{S}}H] < \infty \textit{ for all } \mathcal{S} \in \mathcal{L}_n \textit{ with } |\mathcal{S}| = k\} \\
&= \min\{k : [D/H : \theta(D_{\mathcal{S}})] < \infty \textit{ for all } \mathcal{S} \in \mathcal{L}_n \textit{ with } |\mathcal{S}| = k\}.
\end{aligned}$$

Proof. That the two integers defined in the lemma are equal follows from the fact that $[D : D_{\mathcal{S}}H] = [D/H : D_{\mathcal{S}}H/H] = [D/H : \theta(D_{\mathcal{S}})]$. To see that these are equal to the depth of H, note that, for any $\mathcal{S} \in \mathcal{L}_n$, $[D : D_{\mathcal{S}}H] = [D/D_{\mathcal{S}} : D_{\mathcal{S}}H/D_{\mathcal{S}}] = [D_{\mathcal{S}'} : p_{\mathcal{S}'}(H)]$, where $\mathcal{S}'$ is the complement of $\mathcal{S}$ in $\{1, \dots, n\}$. Thus $D_{\mathcal{S}}H$ has finite index in D for all $\mathcal{S} \in \mathcal{L}_n$ with $|\mathcal{S}| = k$ if and only if $p_{\mathcal{S}}(H)$ has finite index in $D_{\mathcal{S}}$ for all $\mathcal{S} \in \mathcal{L}_n$ with $|\mathcal{S}| = n - k$. □

12.2 Depth 1 subgroups

The following result is essentially contained in [13, Theorem 4.7].

Proposition 12.3. *Let H be a depth 1 subgroup of a direct product $D = \Gamma_1 \times \dots \times \Gamma_n$, where $n \geq 3$. Then H is virtually-coabelian.*

Proof. Since H has depth 1, $[D : \Gamma_i H] < \infty$ for each i. Define D' to be the finite-index subgroup $\cap_{i=1}^n \Gamma_i H \leq D$ and, for each i, define $\Gamma_i' = \Gamma_i \cap D'$. Then, for each k, $\Gamma_k' H = (\Gamma_k \cap D')H = \Gamma_k H \cap D' = D'$. Thus H is full in D'.

We will show that H is coabelian in D' by demonstrating that, for each i, $[\Gamma_i', \Gamma_i'] \leq H$. Given $i \in \{1, \dots, n\}$, choose $j, k \in \{1, \dots, n\}$ so that i, j, k are pairwise distinct. Then, given $\gamma_1, \gamma_2 \in \Gamma_i'$, there exist $g_1 \in \Gamma_j'$, $g_2 \in \Gamma_k'$ and $h_1, h_2 \in H$ so that $\gamma_1 = g_1 h_1$ and $\gamma_2 = g_2 h_2$. Thus $[\gamma_1, \gamma_2] = [\gamma_1 g_1^{-1}, \gamma_2 g_2^{-1}] = [h_1, h_2] \in H$. □

We thus have the following corollary to Theorem 11.3. Note that Part (2) of this result was first proved by Bridson, Howie, Miller and Short [17], but our proof is independent of theirs.

Corollary 12.4. *Let H be a depth 1 subgroup of a direct product $D = \Gamma_1 \times \dots \times \Gamma_n$, where $n \geq 3$.*

(1) If each Γ_i is finitely generated then H is finitely generated and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.

(2) If each Γ_i is finitely presented then H is finitely presented.

(3) If, furthermore, for each i, there exist polynomials α_i and ρ_i such that (α_i, ρ_i) is an area-radius pair for some finite presentation of Γ_i, then H satisfies a polynomial isoperimetric inequality.

12.3 Subdirect products of limit groups

The following conjecture, for which the author of this thesis makes no claims of ownership, has been suggested by various people.

Conjecture 12.5. *Let $L_1, \dots, L_n$ be $n \geq 2$ non-abelian limit groups and let H be a subdirect product of $D = L_1 \times \dots \times L_n$ that intersects each factor non-trivially. Let k be an integer ≥ 2. Then the following are equivalent:*

(1) H is of type F_k;

(2) H is of type $\mathrm{FP}_k(\mathbb{Q})$;

(3) $H_i(H'; \mathbb{Q})$ has finite $\mathbb{Q}$-dimension for all $i \leq k$ and all finite-index subgroups $H' \leq H$;

(4) $\operatorname{Depth} H \leq n-k$.

Note that it is easy to construct examples demonstrating that each of the 3 conditions (H being subdirect; each L_i being non-abelian; and each intersection $H \cap L_i$ being non-trivial) are necessary for depth to be related to finiteness in this way.

Various results provide corroborating evidence for Conjecture 12.5. It is standard that (1) implies (2) implies (3). Meinert [32] has proved that if the L_i are free and H is coabelian in D then conditions (1), (2) and (4) are equivalent. Bridson, Howie, Miller and Short [17] have proved that, in the $k=2$ case, the conditions (1), (2) and (4) are equivalent. It then follows from standard results that (1) and (2) are equivalent in all cases. Building on this work, Kochloukova [31] has proved that condition (3) implies condition (4); and that (3) and (4) are equivalent under certain stronger hypotheses.

We have the following corollary to Kochloukova's result.

Corollary 12.6. *Let $L_1, \ldots, L_n$ be non-abelian limit groups, with $n \geq 3$, and let H be a subdirect product of $D = L_1 \times \ldots \times L_n$ that intersects each factor L_i non-trivially. Suppose that H is of type* $\mathrm{FP}_{n-1}(\mathbb{Q})$. *Then H is finitely presented and satisfies a polynomial isoperimetric inequality, and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.*

Proof. Since H is of type $\mathrm{FP}_{n-1}(\mathbb{Q})$, [31, Theorem 7] implies that H has depth 1 in D. The result then follows from Corollary 12.4 on noting that, since limit groups are CAT(0) [1], they admit a quadratic-linear area-radius pair [16, Proposition III.Γ.1.6]. ☐

Proposition 12.7. *Let $L_1, \ldots, L_n$ be limit groups and let H be a finitely generated subgroup of the direct product $D = L_1 \times \ldots \times L_n$. Then there exist non-abelian limit groups $L'_1, \ldots, L'_{n'}$, with $n' \leq n$, and there exists a subdirect product $H' \leq D' = L'_1 \times \ldots \times L'_{n'}$ with each intersection $L'_i \cap H'$ non-trivial, so that $H' \times A'$ is isomorphic to a finite index subgroup of H for some finitely generated free abelian group A'. Furthermore, if Δ and Δ' are the distortion functions of H in D and H' in D' respectively, then $\Delta(l) \preccurlyeq \Delta'(l) + l$.*

Proof. If one of the intersections $L_i \cap H$ is trivial then the projection homomorphism $q_i : D \to L_1 \times \ldots \times L_{i-1} \times L_{i+1} \times \ldots \times L_n$ is injective on H. Thus H is isomorphic to a subgroup $q_i(H) \leq L_1 \times \ldots \times L_{i-1} \times L_{i+1} \times \ldots \times L_n$ and, by Lemma 4.5, the distortion of H in D is at most the distortion of $q_i(H)$ in $q_i(D)$. Thus, without loss of generality, we may assume that each of the intersections $L_i \cap H$ is non-trivial.

For each i, let $p_i : D \to L_i$ be the projection homomorphism onto the factor L_i. Since H is finitely generated, each $p_i(H)$ is finitely generated and is thus a limit group. By [42, Corollary 3.12], $p_i(H)$ is undistorted in L_i. Thus, by Lemma 4.3 (1), we may assume that H projects onto each L_i.

If all of the L_i are non-abelian then the proposition is proved. Otherwise, define A to be the direct product of those L_i which are abelian, and let $L'_1, \ldots, L'_{n'}$ be those L_i (in some order) which are non-abelian. Define $D' = L'_1 \times \ldots \times L'_{n'}$. Then A is finitely generated free abelian and H is a subdirect product of $D' \times A$ with each intersection $L'_i \cap H$ non-trivial and the intersection $A \cap H$ non-trivial. Since A is finitely generated free abelian, $A \cap H$ is a direct factor of some finite-index subgroup $\bar{A} \leq A$. Define K to be the finite-index subgroup $(D' \times \bar{A}) \cap H$

of H and note that $K \leq D' \times \bar{A}$ and that $\bar{A} \cap K = A \cap H$ is a direct factor of $\bar{A}$. Let C be a choice of complement of $\bar{A} \cap K$ in $\bar{A}$ and define λ to be the projection homomorphism $D' \times \bar{A} = D' \times (\bar{A} \cap K) \times C \to D' \times (\bar{A} \cap K)$. Note that λ is injective on K and that $\lambda(K) = H' \times A'$ where H' is the image of K under the projection $D' \times \bar{A} \to D'$ and $A' = \bar{A} \cap K$.

Given a pair of finitely generated groups $G_1 \leq G_2$, we write $\Delta_{G_1}^{G_2}$ for the distortion function of G_1 in G_2 (defined up to $\approx$-equivalence). Then, applying Corollary 4.4 and Lemma 4.5, we have that $\Delta = \Delta_H^D \approx \Delta_K^{D' \times \bar{A}} \preccurlyeq \Delta_{\lambda(K)}^{D' \times (\bar{A} \cap K)} = \Delta_{H' \times A'}^{D' \times A'} \approx \Delta_{H'}^{D'} + \Delta_{A'}^{A'} = \Delta' + \Delta_{A'}^{A'}$. Thus the proof is complete on noting that for any group G, $\Delta_G^G(l) = l$. ☐

Lemma 12.8. *Let H be a subgroup of a direct product D of at most 2 limit groups and suppose that H is of type $\mathrm{FP}_2(\mathbb{Q})$. Then H is finitely presented, satisfies a quadratic isoperimetric inequality, and is undistorted in D.*

Proof. Since limit groups are CAT(0) [1] they admit quadratic isoperimetric functions [16, Proposition III.Γ.1.6]. Thus D admits a quadratic isoperimetric function. By [20, Lemma 7], H is a virtual retract of D and so the result follows immediately. ☐

Theorem 12.9. *Let $L_1, \ldots, L_n$ be limit groups and let H be a subgroup of the direct product $D = L_1 \times \ldots \times L_n$. Suppose that H is of type $\mathrm{FP}_m(\mathbb{Q})$, where $m = \max\{2, n-1\}$. Then H is finitely presented and satisfies a polynomial isoperimetric inequality, and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.*

Proof. If A is a finitely generated free abelian group and G is an arbitrary group, then each of the following three group-theoretic properties is possessed by G if and only if it is possessed by $G \times A$: being finitely presented; being of type $\mathrm{FP}_m(\mathbb{Q})$; and satisfying a polynomial isoperimetric inequality. Furthermore, each of these three properties is preserved under passage to finite index subgroups and finite index extensions. The theorem thus follows directly from Corollary 12.6, Proposition 12.7 and Lemma 12.8. ☐

Note that the assertion that a subgroup of a direct product of 3 limit groups that is of type $\mathrm{FP}_2(\mathbb{Q})$ is finitely presented was first obtained by Bridson, Howie, Miller and Short [17].

Corollary 12.10. *Let H be a finitely presented subgroup of a direct product D of at most 3 limit groups. Then H satisfies a polynomial isoperimetric inequality and the distortion function Δ of H in D satisfies $\Delta(l) \preccurlyeq l^2$.*

13 A class of full coabelian subdirect products of free groups

In this section we study a class of full, coabelian subdirect products of free groups that have particularly regular structure. We focus in detail on the member $K_2^3(2)$ of this class; this group is singled out as it is the simplest subdirect product of free groups which is not already well-understood. We derive a finite presentation for $K_2^3(2)$ and prove that its Dehn function δ satisfies $\delta(l) \succeq l^3$.

This is the first known example of a subdirect product of free groups that has Dehn function growing faster than that of the ambient direct product.

13.1 Defining the class

We first fix some notation which will be used throughout the section. Given integers $i, m \in \mathbb{N}$ let $F_m^{(i)}$ be the rank m free group with basis $e_1^{(i)}, \ldots, e_m^{(i)}$. Given an integer $r \in \mathbb{N}$ let $\mathbb{Z}^r$ be the rank r free abelian group with basis $t_1, \ldots, t_r$.

Given positive integers $n, m \geq 1$ and $r \leq m$ we wish to define a group $K_m^n(r)$ to be the kernel of a homomorphism $\theta : F_m^{(1)} \times \ldots \times F_m^{(n)} \to \mathbb{Z}^r$ whose restriction to each factor $F_m^{(i)}$ is surjective. For fixed n, m and r, the isomorphism class of the group $K_m^n(r)$ is, up to an automorphism of the factors of the ambient group $F_m^{(1)} \times \ldots \times F_m^{(n)}$, independent of the homomorphism θ. This is proved by the following lemma.

Lemma 13.1. *Let F be a rank m free group. Given a surjective homomorphism $\phi : F \to \mathbb{Z}^r$ there exists a basis $e_1, \ldots, e_m$ of F so that*

$$\phi(e_i) = \begin{cases} t_i & \text{if } 1 \leq i \leq r \\ 0 & \text{if } r+1 \leq i \leq m. \end{cases}$$

Proof. ϕ factors through the abelianisation homomorphism $\mathrm{Ab} : F \to A$, where A is the rank m free abelian group $F/[F,F]$, as $\phi = \bar{\phi} \circ \mathrm{Ab}$ for some homomorphism $\bar{\phi} : A \to \mathbb{Z}^r$. Since $\bar{\phi}$ is surjective A splits as $A_1 \oplus A_2$ where $\bar{\phi}$ is an isomorphism on the first factor and 0 on the second factor. There thus exists a basis $s_1, \ldots, s_m$ for A so as

$$\bar{\phi}(s_i) = \begin{cases} t_i & \text{if } 1 \leq i \leq r \\ 0 & \text{if } r+1 \leq i \leq m. \end{cases}$$

We claim that the s_i lift under Ab to a basis for F. To see this let $f_1, \ldots, f_m$ be any basis for F and let $\bar{f}_1, \ldots, \bar{f}_m$ be its image under Ab, a basis for A. Let $\rho \in \mathrm{Aut}(A)$ be the change of basis isomorphism from $\bar{f}_1, \ldots, \bar{f}_m$ to $s_1, \ldots, s_m$. It suffices to show that this lifts under Ab to an automorphism of F. But this is certainly the case since $\mathrm{Aut}(A) \cong GL_m(\mathbb{Z})$ is generated by the elementary transformations and each of these obviously lifts to an automorphism. $\square$

Definition 13.2. For integers $n, m \geq 1$ and $r \leq m$ define $K_m^n(r)$ to be the kernel of the homomorphism $\theta : F_m^{(1)} \times \ldots \times F_m^{(n)} \to \mathbb{Z}^r$ given by

$$\theta(e_j^{(i)}) = \begin{cases} t_j & \text{if } 1 \leq j \leq r \\ 0 & \text{if } r+1 \leq j \leq m. \end{cases}$$

Note that $K_2^n(1)$ is the n^{th} Stallings-Bieri group SB_n. By a result in Section 1.6 of [32], if $r \geq 1$ and $m \geq 2$ then $K_m^n(r)$ is of type F_{n-1} but not of type FP_n.

Proposition 13.3.

(1) If $n \geq 2$, then $K_m^n(r)$ is finitely generated and has distortion function Δ in $F_m^{(1)} \times \ldots \times F_m^{(n)}$ satisfying $\Delta(l) \preccurlyeq l^2$.

(2) *If $n \geq 3$, then $K^n_m(r)$ is finitely presented and has Dehn function δ satisfying $\delta(l) \preccurlyeq l^{2+2r}$.*

(3) *If $n \geq \{3, 2r\}$, then $K^n_m(r)$ is finitely presented and has Dehn function δ satisfying $\delta(l) \preceq l^5$.*

Proof. This follows immediately from Theorem 11.3. For (2), note that a finitely generated free group admits an area-radius pair (α, ρ) with α and ρ linear. For (3), note that a direct products of finitely generated free groups has Dehn function d satisfying $d(l) \leq Cl^2$, for some $C \in \mathbb{N}$. □

13.2 A splitting theorem

Given a collection of groups $M, L_1, \ldots, L_r$ with $M \leq L_i$ for each i, we denote by $\mathop{\ast}_{i=1}^{r}(L_i\,;\,M)$ the amalgamated product $L_1 *_M \ldots *_M L_r$.

Theorem 13.4. *If $n \geq 2$ and $r \geq 1$ then*

$$K^n_m(r) \cong \left[\mathop{\ast}\nolimits_{k=1}^{r}(L_k\,;\,M)\right] \mathop{\ast}_{M} \left[M \times F_{m-r}\right]$$

where F_{m-r} is a rank $m-r$ free group, $M = K^{n-1}_m(r)$, and for each $k = 1, \ldots, r$ the group $L_k \cong K^{n-1}_m(r-1)$ is the kernel of the homomorphism

$$\theta_k : F^{(1)}_m \times \ldots \times F^{(n-1)}_m \to \mathbb{Z}^{r-1}$$

given by

$$\theta_k(e^{(i)}_j) = \begin{cases} t_j & \text{if } 1 \leq j \leq k-1, \\ 0 & \text{if } j = k, \\ t_{j-1} & \text{if } k+1 \leq j \leq r, \\ 0 & \text{if } r+1 \leq j \leq m. \end{cases}$$

Proof. Projecting $K^n_m(r)$ onto the factor $F^{(n)}_m$ gives the short exact sequence $1 \to K^{n-1}_m(r) \to K^n_m(r) \to F^{(n)}_m \to 1$. This splits to show that $K^n_m(r)$ has the structure of an internal semidirect product $M \rtimes \hat{F}^{(n)}_m$ where $\hat{F}^{(n)}_m \cong F^{(n)}_m$ is the subgroup of $F^{(n-1)}_m \times F^{(n)}_m$ generated by

$$e^{(n-1)}_1(e^{(n)}_1)^{-1},\ \ldots,\ e^{(n-1)}_r(e^{(n)}_r)^{-1},\ e^{(n)}_{r+1},\ \ldots,\ e^{(n)}_m.$$

Since the action by conjugation of $e^{(n-1)}_k(e^{(n)}_k)^{-1}$ on M is the same as the action of $e^{(n-1)}_k$ and since $e^{(n)}_k$ centralises M we have that

$$\begin{aligned} K^n_m(r) &= M \rtimes \hat{F}^{(n)}_m \\ &= \left[\mathop{\ast}\nolimits_{k=1}^{r}\Big(M \rtimes \Big\langle e^{(n-1)}_k(e^{(n)}_k)^{-1}\Big\rangle\,;\,M\Big)\right] \mathop{\ast}_{M} \left[\mathop{\ast}\nolimits_{k=r+1}^{m}\Big(M \rtimes \Big\langle e^{(n)}_k\Big\rangle\,;\,M\Big)\right] \\ &\cong \left[\mathop{\ast}\nolimits_{k=1}^{r}\Big(M \rtimes \Big\langle e^{(n-1)}_k\Big\rangle\,;\,M\Big)\right] \mathop{\ast}_{M} \left[\mathop{\ast}\nolimits_{k=1}^{m-r}\Big(M \times \mathbb{Z}\,;\,M\Big)\right] \\ &\cong \left[\mathop{\ast}\nolimits_{k=1}^{r}\Big(M \rtimes \Big\langle e^{(n-1)}_k\Big\rangle\,;\,M\Big)\right] \mathop{\ast}_{M} \left[M \times F_{m-r}\right]. \end{aligned}$$

Define a homomorphism $p_k : F_m^{(1)} \times \ldots \times F_m^{(n-1)} \to \mathbb{Z}$ by

$$p_k\left(e_j^{(i)}\right) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and note that $L_k \cap \ker p_k$ is the kernel of the standard homomorphism $\theta : F_m^{(1)} \times \ldots \times F_m^{(n-1)} \to \mathbb{Z}^r$ given in definition 13.2. Considering the restriction of p_k to L_k gives the short exact sequence $1 \to K_m^{n-1}(r) \to L_k \to \mathbb{Z} \to 1$ which demonstrates that $L_k = K_m^{(n-1)}(r) \rtimes \langle e_k^{(n-1)} \rangle$. □

Note that as a special case of this proposition we obtain

$$\mathrm{SB}_3 = K_2^3(1) \cong K_2^2(0) *_{K_2^2(1)} (K_2^2(1) \times \mathbb{Z}) \cong (F_2 \times F_2) \dot{*}_{K_2^2(1)}$$

where $\dot{*}$ denotes the trivial HNN extension with amalgamating homomorphism the identity. This yields the presentation of Stallings' group used in [27].

13.3 Generating sets

We give finite generating sets for those groups $K_m^n(r)$ which are finitely generated.

Proposition 13.5. *If $n \geq 2$ then $K_m^n(r)$ is generated by $S_1 \cup S_2 \cup S_3$ where*

$$\begin{aligned} S_1 &= \{e_i^{(1)}(e_i^{(k)})^{-1} \,:\, 1 \leq i \leq r, 2 \leq k \leq n\}, \\ S_2 &= \{e_i^{(k)} \,:\, r+1 \leq i \leq m, 1 \leq k \leq n\}, \\ S_3 &= \{[e_i^{(1)}, e_j^{(1)}] \,:\, 1 \leq i < j \leq r\}. \end{aligned}$$

If $n \geq 3$ then $K_m^n(r)$ is generated by $S_1 \cup S_2$.

Proof. Fix $n \geq 2$, $m \geq 1$ and $r \leq m$. Let θ be the homomorphism given in Definition 13.2. Since $n \geq 2$, $K_m^n(r)$ is the fibre product of the homomorphisms $\theta|_{F_m^{(1)}}$ and $-\theta|_{F_m^{(2)} \times \ldots \times F_m^{(1)}}$. Define the following collections of elements of $K_m^n(r)$:

$$\begin{aligned} \mathcal{T}_1 &= \{e_i^{(1)}(e_i^{(2)})^{-1} : 1 \leq i \leq r\} \cup \{e_i^{(1)} : r+1 \leq i \leq m\}; \\ \mathcal{T}_2 &= \{e_i^{(2)}(e_i^{(k)})^{-1} : 1 \leq i \leq r, 3 \leq k \leq n\} \cup \{e_i^{(k)} : r+1 \leq i \leq m, 2 \leq k \leq n\}; \\ \mathcal{T}_3 &= \{[e_i^{(1)}, e_j^{(1)}] : 1 \leq i < j \leq r\}. \end{aligned}$$

By Lemma 9.5, $K_m^n(r)$ is generated by $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Now note that each element of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ can be expressed in terms of the $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

If $n \geq 3$ then $S_1 \cup S_2$ suffices since as group elements $[e_i^{(1)}, e_j^{(2)}] = [e_i^{(1)}(e_i^{(2)})^{-1}, e_j^{(1)}(e_j^{(3)})^{-1}]$. □

13.4 A presentation for $K_2^3(1)$

In Sections 13.4 and 13.5 we derive finite presentations for the groups $K_m^n(r)$ in the case $m = 2, n = 3$. To simplify notation we write x_i for $e_1^{(i)}$ and y_i for $e_2^{(i)}$. Note that we have a short exact sequence $1 \to K_2^3(2) \to K_2^3(1) \to \mathbb{Z} \to 1$, where the homomorphism $K_2^3(1) \to \mathbb{Z}$ is given by mapping each $x_i \mapsto 0$ and

each $y_i \mapsto 1$. Finite presentations for $K_2^3(1)$ have been derived elsewhere; we derive a presentation in positive normal form with respect to the above short exact sequence, so as we can apply Theorem 7.1 to derive a presentation for $K_2^3(2)$.

Let $\alpha_1 = x_1 x_2^{-1}$, $\alpha_2 = x_1 x_3^{-1}$, $\beta_1 = y_1 y_2^{-1}$, $\beta_2 = y_1 y_3^{-1}$ and $t = y_1$. Define $\mathcal{R}$ to be the collection of relations:

$$\begin{array}{ccc} [\alpha_1, \alpha_2] & [\alpha_1, \beta_2][\alpha_2, \beta_1]^{-1} & [\alpha_1, \beta_2^{-1}][\alpha_2, \beta_1^{-1}]^{-1} \\ [\beta_1, \beta_2] & [\alpha_1^{-1}, \beta_2][\alpha_2^{-1}, \beta_1]^{-1} & [\alpha_1^{-1}, \beta_2^{-1}][\alpha_2^{-1}, \beta_1^{-1}]^{-1} \end{array}$$

Proposition 13.6. *Each of the following presents $K_2^3(1)$:*

$$\mathcal{P}_1 = \langle \alpha_1, \alpha_2, y_1, y_2, y_3 \,|\, [\alpha_1, \alpha_2],\ [y_1, y_2],\ [y_1, y_3],\ [y_2, y_3],\ [\alpha_1, y_3],\ [\alpha_2, y_2],\ [\alpha_1^{-1}\alpha_2, y_1] \rangle$$
$$\mathcal{P}_2 = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, t \,|\, [\alpha_1, \alpha_2],\ [\beta_1, \beta_2],\ [t, \beta_1], [t, \beta_2], [\alpha_1, t\beta_2^{-1}],\ [\alpha_2, t\beta_1^{-1}], [\alpha_1^{-1}\alpha_2, t] \rangle$$
$$\mathcal{P}_3 = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, t \,|\, \mathcal{R},\ [t, \beta_1],\ [t, \beta_2],\ \alpha_1^t = \alpha_1^{\beta_2},\ \alpha_2^t = \alpha_2^{\beta_1} \rangle$$

Proof. That the stated elements generate follows from Proposition 13.5. The proof that the relations in presentation $\mathcal{P}_1$ suffice is almost identical to a proof given by Gersten [27], who derives a presentation of the group $\ker(F_2^{(1)} \times F_2^{(2)} \times F_2^{(3)} \to \mathbb{Z})$ where the homomorphism maps each of the chosen basis elements of $F_2^{(i)}$ to the chosen generator of $\mathbb{Z}$. We briefly recount the argument.

Let $w \equiv w(\alpha_1, \alpha_2, y_1, y_2, y_3)$ be a null-homotopic word in $K_2^3(1)$. Note that w is freely equal to a word $w'(\alpha_1, \alpha_2, y_2, y_3) \prod_{i=1}^k y_1^{\epsilon_i w_i(\alpha_1, \alpha_2, y_2, y_3)}$ for some words w' and w_i and some $\epsilon_i \in \{\pm 1\}$, and that the relations $[\alpha_1, \alpha_2]$, $[\alpha_1, y_3]$, $[\alpha_2, y_2]$ and $[y_2, y_3]$ are sufficient to convert this to a word of the form

$$u(\alpha_1, y_2) v(\alpha_2, y_3) \prod_{i=1}^k y_1^{\epsilon_i u_i(\alpha_1, y_2) v_i(\alpha_2, y_3)}$$

for some words u, u_i and v_i. The relation $[\alpha_1^{-1}\alpha_2, y_1]$ is equivalent to $y_1^{\alpha_1} = y_1^{\alpha_2}$ and this, together with the relations $[y_1, y_2]$ and $[\alpha_2, y_2]$, are sufficient to convert the above word to a word $u(\alpha_1, y_2) v(\alpha_2, y_3) \prod_{i=1}^k y_1^{\epsilon_i v_i'(\alpha_2, y_3)}$ for some words v_i'. Finally this can be converted to a word $u(\alpha_1, y_2) v(\alpha_2, y_3) \prod_{i=1}^k y_1^{\epsilon_i \alpha_1^{n_i}}$, where the $n_i \in \mathbb{Z}$, by applying the relations $[\alpha_1^{-1}\alpha_2, y_1]$, $[\alpha_1, y_3]$ and $[y_1, y_3]$.

As a group element this word is equal to

$$u(x_1, \emptyset) v(x_1, \emptyset) u(x_2^{-1}, y_2) v(x_3^{-1}, y_3) \prod_{i=1}^k x_1^{n_i} y_1^{\epsilon_i} x_1^{-n_i}.$$

Since $\{x_2^{-1}, y_2\}$ and $\{x_3^{-1}, y_3\}$ form free bases for $F_2^{(2)}$ and $F_3^{(3)}$ respectively it must be that u and v are freely equal to the empty word. Similarly the elements $\{x_1^n y_1 x_1^{-n} \,:\, n \in \mathbb{Z}\}$ are freely independent so the product term also freely reduces to the empty word. This completes the proof that $\mathcal{P}_1$ presents $K_2^3(1)$.

To show that presentations $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent, substitute $t = y_1$, $\beta_1 = t y_2^{-1}$ and $\beta_2 = t y_3^{-1}$ into $\mathcal{P}_1$ to give the presentation

$$\langle \alpha_1, \alpha_2, \beta_1, \beta_2, t \,|\, [\alpha_1, \alpha_2],\ [t, \beta_1^{-1}t],\ [t, \beta_2^{-1}t],\ [\beta_1^{-1}t, \beta_2^{-1}t],\ [\alpha_1, \beta_2^{-1}t],\ [\alpha_2, \beta_1^{-1}t],\ [\alpha_1^{-1}\alpha_2, t] \rangle$$

which can easily be converted to $\mathcal{P}_2$.

Finally, we show that the presentations $\mathcal{P}_2$ and $\mathcal{P}_3$ are Tietze equivalent. The van Kampen diagram in Figure 4 (together with three similar ones) demonstrates that the relations in $\mathcal{R}$ are null-homotopic over $\mathcal{P}_2$. Conversely, the van Kampen diagram in Figure 5 demonstrates that the relation $[\alpha_1^{-1}\alpha_2, t]$ is null-homotopic over presentation $\mathcal{P}_3$.

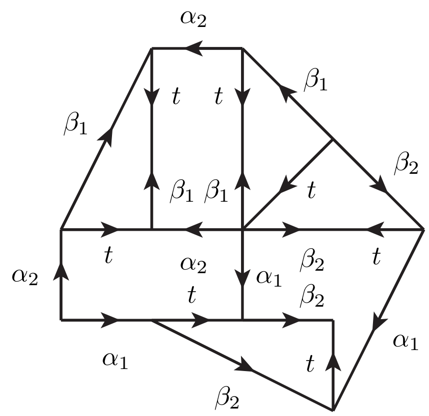

Figure 4: $\mathcal{P}_2$-van Kampen diagram for $[\alpha_1, \beta_2][\alpha_2, \beta_1]^{-1}$

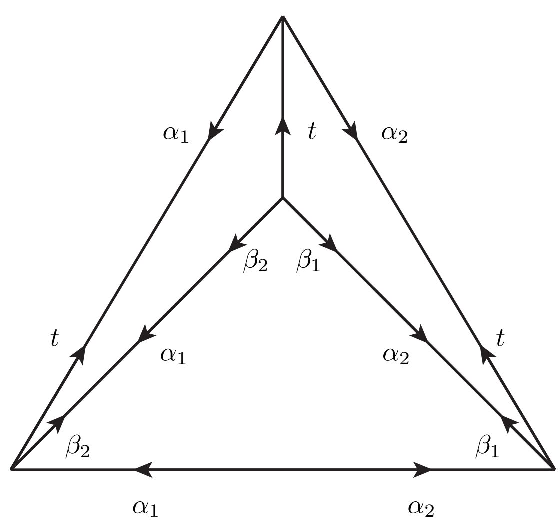

Figure 5: $\mathcal{P}_3$-van Kampen diagram for $[\alpha_1^{-1}\alpha_2, t]$

□

13.5 A presentation for $K_2^3(2)$

By Proposition 13.5, the group $K_2^3(2)$ is generated by $\mathcal{X} = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. Define $\mathcal{R}_1$ to be the collection of relations $\mathcal{R}$, which we recall here for ease of use:

$$\begin{array}{ccc} [\alpha_1,\alpha_2] & [\alpha_1,\beta_2][\alpha_2,\beta_1]^{-1} & [\alpha_1,\beta_2^{-1}][\alpha_2,\beta_1^{-1}]^{-1} \\ [\beta_1,\beta_2] & [\alpha_1^{-1},\beta_2][\alpha_2^{-1},\beta_1]^{-1} & [\alpha_1^{-1},\beta_2^{-1}][\alpha_2^{-1},\beta_1^{-1}]^{-1} \end{array}$$

Define $\mathcal{R}_2$ to be the collection of relations:

$$\begin{array}{cccc} [\alpha_1,\alpha_2] & [\beta_2^{\alpha_1},\beta_2^{-1}\beta_1] & [\alpha_2^{\beta_1},\alpha_2^{-1}\alpha_1] & \\ & & & [\alpha_1,\beta_2][\alpha_2,\beta_1]^{-1} \\ [\beta_1,\beta_2] & [\beta_2^{\alpha_1^{-1}},\beta_2^{-1}\beta_1] & [\alpha_2^{\beta_1^{-1}},\alpha_2^{-1}\alpha_1] & \end{array}$$

Proposition 13.7. *The group $K_2^3(2)$ is presented by both $\mathcal{Q}_1 = \langle \mathcal{X} \mid \mathcal{R}_1 \rangle$ and $\mathcal{Q}_2 = \langle \mathcal{A} \mid \mathcal{R}_2 \rangle$.*

Proof. We first prove that $\mathcal{Q}_1$ presents $K_2^3(2)$. For each $x \in \mathcal{X}$, define words $w_x^+, w_x^- \in \mathcal{X}^{\pm *}$ as in the following table.

$x \in \mathcal{X}$	w_x^+	w_x^-
α_1	$\beta_2\alpha_1\beta_2^{-1}$	$\beta_2^{-1}\alpha_1\beta_2$
α_2	$\beta_1\alpha_2\beta_1^{-1}$	$\beta_1^{-1}\alpha_2\beta_1$
β_1	β_1	β_1
β_2	β_2	β_2

Define Φ^+, Φ^- and $\mathcal{S}^+$, $\mathcal{S}^-$ as in the preamble to Theorem 7.1. By Proposition 13.6, $K_2^3(1)$ is presented by $\langle \mathcal{X}, t \mid \mathcal{R}_1, \mathcal{S}^+ \rangle$. The relations $\mathcal{S}^-$ are (easy) consequences of the relations $\mathcal{R}_1 \cup \mathcal{S}^+$ and so $K_2^3(1)$ is also presented by $\langle \mathcal{X}, t \mid \mathcal{R}_1, \mathcal{S}^+, \mathcal{S}^- \rangle$. We are thus in a position to apply Theorem 7.1.

For each $x \in \mathcal{X}$, the relation $x\Phi^-(\Phi^+(x))$ is freely trivial. It thus suffices to show that all the words $\Phi^\epsilon(r)$, where $\epsilon \in \{\pm 1\}$ and $r \in \mathcal{R}_1$ are null-homotopic over $\mathcal{P}_1$. These relations are given in the following table.

$r \in \mathcal{R}_1$	$\Phi^+(r)$	$\Phi^-(r)$
$[\alpha_1,\alpha_2]$	$[\alpha_1^{\beta_2},\alpha_2^{\beta_1}]$	$[\alpha_1^{\beta_2^{-1}},\alpha_2^{\beta_1^{-1}}]$
$[\beta_1,\beta_2]$	$[\beta_1,\beta_2]$	$[\beta_1,\beta_2]$
$[\alpha_1,\beta_2][\alpha_2,\beta_1]^{-1}$	$[\alpha_1^{\beta_2},\beta_2][\alpha_2^{\beta_1},\beta_1]^{-1}$	$[\alpha_1^{\beta_2^{-1}},\beta_2][\alpha_2^{\beta_1^{-1}},\beta_1]^{-1}$
$[\alpha_1^{-1},\beta_2][\alpha_2^{-1},\beta_1]^{-1}$	$[\alpha_1^{-\beta_2},\beta_2][\alpha_2^{-\beta_1},\beta_1]^{-1}$	$[\alpha_1^{-\beta_2^{-1}},\beta_2][\alpha_2^{-\beta_1^{-1}},\beta_1]^{-1}$
$[\alpha_1,\beta_2^{-1}][\alpha_2,\beta_1^{-1}]^{-1}$	$[\alpha_1^{\beta_2},\beta_2^{-1}][\alpha_2^{\beta_1},\beta_1^{-1}]^{-1}$	$[\alpha_1^{\beta_2^{-1}},\beta_2^{-1}][\alpha_2^{\beta_1^{-1}},\beta_1^{-1}]^{-1}$
$[\alpha_1^{-1},\beta_2^{-1}][\alpha_2^{-1},\beta_1^{-1}]^{-1}$	$[\alpha_1^{-\beta_2},\beta_2^{-1}][\alpha_2^{-\beta_1},\beta_1^{-1}]^{-1}$	$[\alpha_1^{-\beta_2^{-1}},\beta_2^{-1}][\alpha_2^{-\beta_1^{-1}},\beta_1^{-1}]^{-1}$

□

Define a monoid endomorphism $\Lambda_\alpha : \mathcal{X}^{\pm *} \to \mathcal{X}^{\pm *}$, which commutes with the inversion automorphism, by mapping $\alpha_i \mapsto \alpha_i^{-1}$ and $\beta_i \mapsto \beta_i$. Similarly, define an endomorphism $\Lambda_\beta : \mathcal{X}^{\pm *} \to \mathcal{X}^{\pm *}$ which commutes with the inversion automorphism by mapping $\alpha_i \mapsto \alpha_i$ and $\beta_i \mapsto \beta_i^{-1}$. Note that if $r \in \mathcal{R}_1$, then both $\Lambda_\alpha(r)$ and $\Lambda_\beta(r)$ are cyclic conjugates of relations also in $\mathcal{R}_1$. It follows that if $w \in \mathcal{X}^{\pm *}$ is null-homotopic over $\mathcal{Q}_1$, then so are are $\Lambda_\alpha(r)$ and $\Lambda_\beta(r)$. Taking this symmetry into account, it thus suffices to show that the

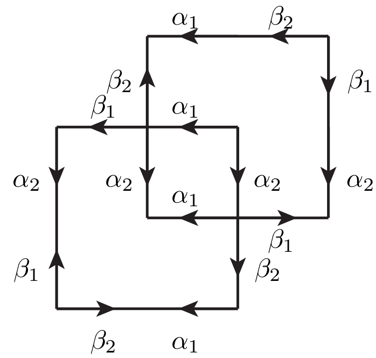

Figure 6: $\mathcal{Q}_1$-van Kampen diagram for $\Phi^+([\alpha_1, \alpha_2])$

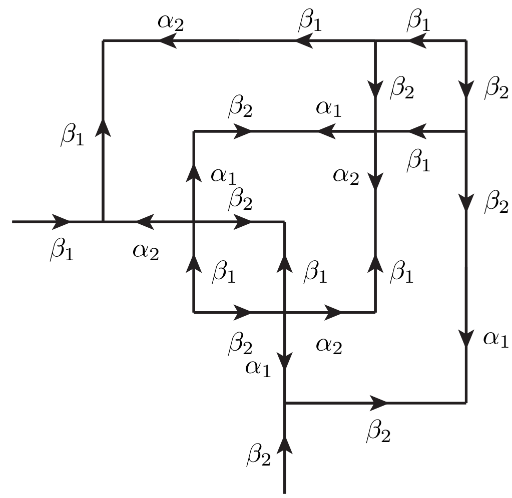

Figure 7: $\mathcal{Q}_1$-van Kampen diagram for $\Phi^+([\alpha_1, \beta_2][\alpha_2, \beta_1]^{-1})$

words $\Phi^+([\alpha_1, \alpha_2])$, $\Phi^+([\alpha_1, \beta_2][\alpha_2, \beta_1]^{-1})$ and $\Phi^-([\alpha_1, \beta_2][\alpha_2, \beta_1]^{-1})$ are null-homotopic over $\mathcal{Q}_1$. $\mathcal{Q}_1$-van Kampen diagrams for these words are displayed in Figures 6, 7 and 8.

Finally, we show that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ define the same group. Define a monoid endomorphism $\overline{\Lambda} : \mathcal{X}^{\pm *} \to \mathcal{X}^{\pm *}$, commuting with the inversion automorphism, by mapping $\alpha_i \mapsto \beta_i$ and $\beta_i \mapsto \alpha_i$. Note that, for $i = 1$ or 2, if r is a relation in $\mathcal{R}_i$, then $\overline{\Lambda}(r)$ is a cyclic conjugate of some relation also in $\mathcal{R}_i$. We show that each of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is Tietze equivalent to the presentation $\langle \mathcal{X} \mid \mathcal{R}_1, \mathcal{R}_2 \rangle$. For the first equivalence, note that $\mathcal{R}_2$ contains 4 relations distinct from those in $\mathcal{R}_1$. Taking into account the symmetries Λ_α, Λ_β and $\overline{\Lambda}$, it suffices to show that the word $[\beta_2^{\alpha_1}, \beta_2^{-1}\beta_1]$ is null-homotopic over $\mathcal{Q}_1$. A $\mathcal{Q}_1$-van Kampen diagram for this word is displayed in Figure 9. For the other equivalence, note that $\mathcal{R}_1$ contains 3 relations distinct from those in $\mathcal{R}_2$. Taking into account the symmetry $\overline{\Lambda}$, it suffices to show that the words $[\alpha_1^{-1}, \beta_2][\alpha_2^{-1}, \beta_1]^{-1}$ and $[\alpha_1^{-1}, \beta_2^{-1}][\alpha_2^{-1}, \beta_1^{-1}]^{-1}$ are null-homotopic over $\mathcal{Q}_2$. $\mathcal{Q}_2$-van Kampen diagrams for these words are displayed in Figures 10 and 11.

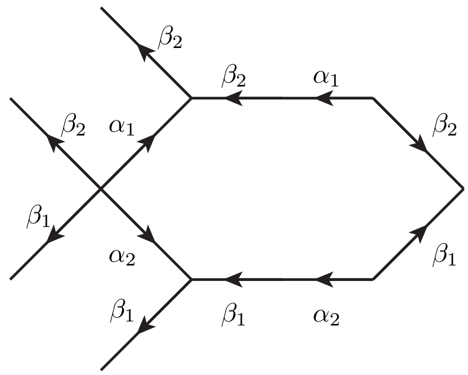

Figure 8: $\mathcal{P}_3$-van Kampen diagram for $\Phi^-([\alpha_1, \beta_2][\alpha_2, \beta_1]^{-1})$

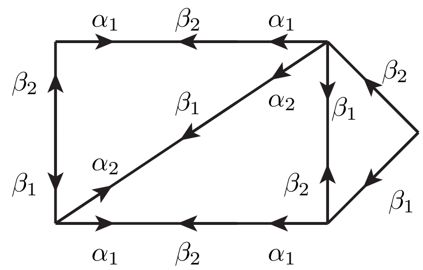

Figure 9: $\mathcal{Q}_1$-van Kampen diagram for $[\beta_2^{\alpha_1}, \beta_2^{-1}\beta_1]$

13.6 A lower bound on the Dehn function of $K_2^3(2)$

Theorem 13.8. *The Dehn function δ of $K_2^3(2)$ satisfies $\delta(l) \succeq l^3$.*

Proof. By Proposition 13.5 and Theorem 13.4, we have that $K_2^3(2) \cong L_1 *_M L_2$ where, as subgroups of $F_2^{(1)} \times F_2^{(2)}$, $L_1 = K_2^2(1)$ is generated by $\mathcal{A}_1 = \{x_1x_2^{-1}, y_1, y_2\}$, $L_2 \cong K_2^2(1)$ is generated by $\mathcal{A}_2 = \{x_1, x_2, y_1y_2^{-1}\}$ and $M = K_2^2(2)$ is generated by $\mathcal{B} = \{x_1x_2^{-1}, y_1y_2^{-1}, [x_1, y_1]\}$. To obtain the generating set for L_2 we have here implicitly used the automorphism of $F_2^{(1)} \times F_2^{(2)}$ which interchanges x_i with y_i and realises the isomorphism between L_2 and $K_2^2(1)$.

For each $l \in \mathbb{N}$, define h_l to be the element $[x_1^l, y_1^l] \in K_2^2(2)$ and define w_l to be the word $[(x_1x_2^{-1})^l, y_1^l] \in \mathcal{A}_1^{\pm *}$ representing h_l. Note that h_l commutes with both $y_2 \in \mathcal{A}_1$ and $x_2 \in \mathcal{A}_2$ so, by Theorem 8.1, the word $[w_l, (y_2x_2)^l]$, which has length $16l$, has area at least $2l\, d_{\mathcal{B}}(1, h_l)$. We claim that $d_{\mathcal{B}}(1, h_l) \geq l^2$.

Suppose that in $F_2^{(1)} \times F_2^{(2)}$ the element h_l is represented by a word $w \equiv w(x_1x_2^{-1}, y_1y_2^{-1}, [x_1, y_1])$ in the generators $\mathcal{B}$. Let k be the number of occurrences of the third variable in the word w. We will show that $k \geq l^2$.

Observe that as group elements the word $w(x_1x_2^{-1}, y_1y_2^{-1}, [x_1, y_1])$ is equal to the word $w(x_1, y_1, [x_1, y_1])\, w(x_2^{-1}, y_2^{-1}, 1)$. Thus we have that $[x_1^l, y_1^l]$ is freely equal to $w(x_1, y_1, [x_1, y_1])$ and that $w(x_2^{-1}, y_2^{-1}, 1)$, and thus $w(x_1, y_1, 1)$, is freely equal to the empty word. It follows that there exists a null $\mathcal{P}$-sequence for $[x_1^l, y_1^l]$ with area k, where $\mathcal{P}$ is the presentation $\langle x_1, y_1 \mid [x_1, y_1] \rangle$. But $\mathcal{P}$ presents the rank 2 free abelian group, and basic results on Dehn functions give that $[x_1^l, y_1^l]$

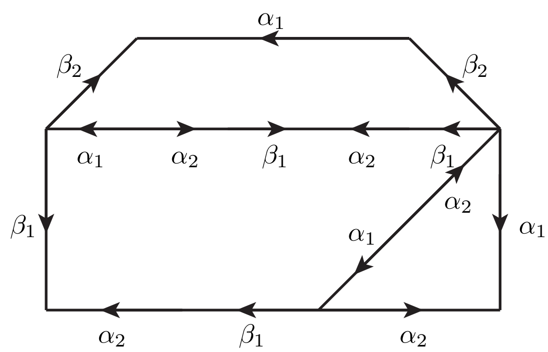

Figure 10: $\mathcal{Q}_2$-van Kampen diagram for $[\alpha_1^{-1}, \beta_2][\alpha_2^{-1}, \beta_1]^{-1}$

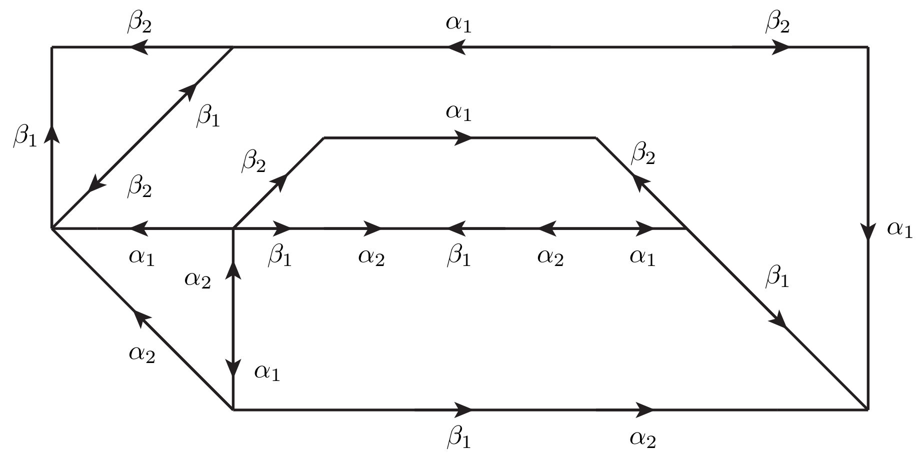

Figure 11: $\mathcal{Q}_2$-van Kampen diagram for $[\alpha_1^{-1}, \beta_2^{-1}][\alpha_2^{-1}, \beta_1^{-1}]^{-1}$

has area l^2 over this presentation. Thus $k \geq l^2$. $\square$

14 Bestvina-Brady groups

Definition 14.1. A simplicial complex is said to be *flag* if every collection of pairwise adjacent vertices spans a simplex. A finite flag simplicial complex Δ with vertices $v_1, \ldots, v_k$ defines an associated *right-angled Artin group* A_Δ given by the presentation

$$\mathcal{P}_A = \langle v_1, \ldots, v_k \mid [v_i, v_j] \text{ whenever } v_i \text{ and } v_j \text{ are joined by an edge in } \Delta \rangle.$$

The *Bestvina-Brady group* H_Δ associated to Δ is defined to be the kernel of the homomorphism $A \to \mathbb{Z} = \langle t \rangle$ which maps each $a_i \mapsto t$.

Definition 14.2. A simplicial complex Δ is said to be n*-connected* (respectively n*-acyclic*), where n is a positive integer, if $\pi_i(\Delta)$ (resp. $H_i(\Delta, \mathbb{Z})$) is trivial for all $i \leq n$.

Theorem 14.3 (Bestvina-Brady [7]).

(1) H_Δ is of type $\mathrm{F_m}$ *if and only if Δ is $(m-1)$-connected.*

(2) H_Δ is of type $\mathrm{FP_m}$ *if and only if Δ is $(m-1)$-acyclic.*

This section is devoted to proving the following result.

Theorem 14.4. *Every finitely presented Bestvina-Brady group has l^4 as an isoperimetric function.*

Theorem 14.4 provides an obstruction to the method suggested in [12] for producing finitely presented Bestvina-Brady groups whose Dehn functions are $\simeq$-equivalent to l^m for arbitrary integers m.

If a Bestvina-Brady group is finitely presented, then Dicks and Leary [23] have shown how to read off from the defining complex a particularly pleasant finite presentation. Let $\mathrm{Edge}(\Delta)$ be the set of directed edges of Δ (so the cardinality of $\mathrm{Edge}(\Delta)$ is twice the number of 1-simplices in Δ). We write ιe and τe respectively for the initial and terminal vertices of e and $\overline{e}$ for the edge e with the opposite orientation. We say that the directed edges $e_1, \ldots, e_n$ form a *combinatorial path* in Δ, written $e_1 \cdot \ldots \cdot e_n$, if $\tau e_i = \iota e_{i+1}$ for all i. If furthermore $\tau e_n = \iota e_1$ then we say that $e_1 \cdot \ldots \cdot e_n$ is a *combinatorial 1-cycle*.

Define $\mathcal{R}_\Delta \subseteq \mathrm{Edge}(\Delta)^{\pm *}$ to consist of all words $e\overline{e}$ for $e \in \mathrm{Edge}(\Delta)$ and all words efg and $e^{-1}f^{-1}g^{-1}$ where $e \cdot f \cdot g$ is a combinatorial 1-cycle in Δ.

Theorem 14.5 (Dicks-Leary [23])**.** *If Δ is simply connected then H_Δ is presented by $\langle \mathrm{Edge}(\Delta) \,|\, \mathcal{R}_\Delta \rangle$ with the embedding $H_\Delta \hookrightarrow A_\Delta$ given by $e \mapsto \iota e (\tau e)^{-1}$.*

The structure of the proof of Theorem 14.4 is as follows. Let H_Δ and A_Δ be the Bestvina-Brady and right-angled Artin groups respectively associated to a simply-connected finite flag simplicial complex Δ. The cyclic extension $1 \to H_\Delta \to A_\Delta \to \mathbb{Z} \to 1$ splits and we take a positive normal form presentation $\langle \mathrm{Edge}(\Delta), t \,|\, \mathcal{R}_\Delta, \mathcal{S}_\Delta \rangle$ for A_Δ, where $\mathcal{P}_H = \langle \mathrm{Edge}(\Delta) \,|\, \mathcal{R}_\Delta \rangle$ is the Dicks-Leary presentation for H_Δ and $\mathcal{S}_\Delta$ consists of a relator of the form $tet^{-1}w_e^{-1}$ with $w_e \in \mathrm{Edge}(\Delta)^{\pm *}$ for each $e \in \mathrm{Edge}(\Delta)$. Since A_Δ is CAT(0) it admits a quadratic-linear area-radius pair [15, Proposition III.Γ.1.6], and so we can apply Theorem 7.4 to produce an infinite indexed presentation $(\mathcal{P}_H^\infty, \|\cdot\|)$ for H_Δ that admits a quadratic-linear area-penetration pair. Lemma 14.15 shows that the relational area function RArea_H of $(\mathcal{P}_H^\infty, \|\cdot\|)$ over $\mathcal{P}_H$ is $\preceq$ quadratic and hence Theorem 14.4 follows by Proposition 6.2. The individual calculations required to prove Lemma 14.15 are set out in Lemmas 14.7–14.14.

Choose a base vertex q and a spanning tree T in the 1-skeleton of Δ. Given $n \in \mathbb{Z}$ and vertices u and v of Δ write $p_n(u,v)$ for the element $e_1^n \ldots e_l^n$ of $\mathrm{Edge}(\Delta)^{\pm *}$ where $e_1 \cdot \ldots \cdot e_l$ is the unique geodesic combinatorial path in T from u to v. We write $p(u,v)$ as shorthand for $p_1(u,v)$. Note that as group elements

$$\begin{aligned} p_n(u,v)^{-1} &= (e_1^n \ldots e_l^n)^{-1} \\ &= e_l^{-n} \ldots e_1^{-n} \\ &= \overline{e_l}^n \ldots \overline{e_1}^n \\ &= p_n(v,u) \end{aligned} \tag{1}$$

in H_Δ. For each $e \in \mathrm{Edge}(\Delta)$, define w_e to be the word $p(q, \iota e) e p(\iota e, q) \in \mathrm{Edge}(\Delta)^{\pm *}$. In [23] it is proved that mapping $e \mapsto w_e$ defines an automorphism θ of H_Δ and that $H_\Delta \rtimes_\theta \mathbb{Z}$ is isomorphic to A_Δ with $e \in \mathrm{Edge}(\Delta)$ corresponding

to $\iota e(\tau e)^{-1}$ and the generator t of $\mathbb{Z}$ corresponding to $q \in A_\Delta$. It is also shown that if $e_1 \cdot \ldots \cdot e_l$ is a combinatorial 1-cycle then $e_1^n \ldots e_l^n$ is null-homotopic in H_Δ. Define $\mathcal{S}_\Delta$ to be the set of words $\{tet^{-1}w_e \,:\, e \in \mathrm{Edge}(\Delta)\} \subseteq (\mathrm{Edge}(\Delta) \cup \{t\})^{\pm *}$. Then A_Δ is finitely presented by $\mathcal{P}'_A = \langle \mathrm{Edge}(\Delta), t \,|\, \mathcal{R}_\Delta, \mathcal{S}_\Delta \rangle$.

The following lemma details some properties of the automorphism θ of H_Δ. Of these we will only need (vii), but this property is most easily proved via the preceding sequence of assertions.

Lemma 14.6. *For all $e \in \mathrm{Edge}(\Delta)$ and $n \in \mathbb{Z}$ the following equalities hold in H_Δ:*

(i) $\theta(e) = p(q, \iota e)ep(q, \iota e)^{-1} = p(q, \iota e)e^2 p(\tau e, q) = p(q, \iota e)e^2 p(q, \tau e)^{-1}$.

(ii) $\theta(e^n) = p(q, \iota e)e^n p(\iota e, q) = p(q, \iota e)e^{n+1} p(\tau e, q) = p(q, \iota e)e^{n+1} p(q, \tau e)^{-1}$.

(iii) If $e_1 \cdot \ldots \cdot e_l$ is a combinatorial path then

$$\theta(e_1^n \ldots e_l^n) = p(q, \iota e_1)e_1^{n+1} \ldots e_l^{n+1} p(\tau e_l, q).$$

(iv) $\theta^{-1}(e) = p_{-1}(q, \iota e)p_{-1}(\tau e, q) = p_{-1}(q, \iota e)ep_{-1}(\iota e, q) = p_{-1}(q, \iota e)ep_{-1}(q, \iota e)^{-1}$.

(v) $\theta^{-1}(e^n) = p_{-1}(q, \iota e)e^n p_{-1}(\iota e, q) = p_{-1}(q, \iota e)e^{n-1} p_{-1}(\tau e, q) = p_{-1}(q, \iota e)e^{n-1} p_{-1}(q, \tau e)^{-1}$.

(vi) If $e_1 \cdot \ldots \cdot e_l$ is a combinatorial path then

$$\theta^{-1}(e_1^n \ldots e_l^n) = p_{-1}(q, \iota e_1)e_1^{n-1} \ldots e_l^{n-1} p_{-1}(\tau e_l, q).$$

(vii) $\theta^k(e) = p_k(q, \iota e)e^{k+1} p_k(\tau e, q)$.

Proof.

(i) The first and third equalities follow from equation (1). The second equality follows from the fact that $p(q, \iota e)ep(\tau e, q)$ is null-homotopic.

(ii) The first equality holds since $\theta(e^n) = \theta(e)^n = [p(q, \iota e)ep(q, \iota e)^{-1}]^n = p(q, \iota e)e^n p(q, \iota e)^{-1} = p(q, \iota e)e^n p(\iota e, q)$ in H_Δ. The second and third equalities then hold since $p(\iota e, q) = ep(\tau e, q)$ in H_Δ and by equation (1) respectively.

(iii) Follows from the fact that $\theta(e_i^n) = p(q, \iota e)e_i^{n+1} p(q, \tau e)^{-1}$ in H_Δ.

(iv) The first equality holds since $\theta(p_{-1}(q, \iota e)p_{-1}(\tau e, q)) = p(q, q)p_0(q, \iota e)p(\iota e, q)p(q, \tau e)p_0(\tau e, q)p(q, q) = p(\iota e, q)p(q, \tau e) = e$ in H_Δ. The second and third equalities follows from the fact that $p_{-1}(q, \tau e)\bar{e}^{-1} p_{-1}(\iota e, q) = p_{-1}(q, \tau e)ep_{-1}(\iota e, q)$ is null-homotopic.

(v) Follows from (iv) as in the proof of (ii).

(vi) Follows from (v) as in the proof of (iii).

(vii) Follows from (iii) and (vi) by induction on $|k|$.

□

For each $n \in \mathbb{Z}$, define a homomorphism $\Phi_n : \mathrm{Edge}(\Delta)^{\pm *} \to \mathrm{Edge}(\Delta)^{\pm *}$ which commutes with the inversion involution and is a lift of θ^n by mapping $e \mapsto p_n(q, \iota e)e^{n+1}p_n(\tau e, q)$. Define the collections of words

$$\overline{\mathcal{R}}_\Delta = \{\Phi_n(r) \,:\, r \in \mathcal{R}_\Delta, n \in \mathbb{Z}\},$$
$$\overline{\mathcal{S}}_\Delta = \{\Phi_{n+1}(e)\Phi_n(w_e)^{-1} \,:\, e \in \mathrm{Edge}(\Delta), n \in \mathbb{Z}\}$$

in $\mathrm{Edge}(\Delta)^{\pm *}$, and consider the presentation $\mathcal{P}_H^\infty = \langle \mathrm{Edge}(\Delta) \,|\, \overline{\mathcal{R}}_\Delta, \overline{\mathcal{S}}_\Delta \rangle$ of H_Δ. Define an index $\|\cdot\|$ on $\overline{\mathcal{R}}_\Delta \cup \overline{\mathcal{S}}_\Delta$ by setting $\|\omega\|$ to be the minimal value of $|k|$ such that either $\omega \equiv \Phi_k(r)$ for some $r \in \mathcal{R}_\Delta$ or $\omega \equiv \Phi_{k+1}(e)\Phi_k(w_e)^{-1}$ for some $e \in \mathrm{Edge}(\Delta)$.

Let d be the length metric on the 1-skeleton of Δ given by setting the length of each edge to 1. Define

$$L = \max\{\mathrm{d}(u, v) \,:\, u, v \in \mathrm{Vert}(\Delta)\}.$$

Lemma 14.7. $\mathrm{Area}_{\mathcal{P}_H}\left(\Phi_n(e\overline{e})\right) \leq (2L+1)|n| + 1$ *for all* $e \in \mathrm{Edge}(\Delta)$.

Proof. The calculation (1) shows that $p_n(q, v)^{-1}$ can be converted to $p_n(v, q)$ at a $\mathcal{P}_H$-cost of at most $L|n|$ for all $v \in \mathrm{Vert}(\Delta)$. The following is a null $\mathcal{P}_H$-scheme for the word $\Phi_n(e\overline{e})$:

j	σ_j	Area
1	$p_n(q, \iota e)e^{n+1}p_n(\tau e, q)p_n(q, \tau e)\overline{e}^{n+1}p_n(\iota e, q)$	$L\|n\|$
2	$p_n(q, \iota e)e^{n+1}\overline{e}^{n+1}p_n(\iota e, q)$	$\|n\| + 1$
3	$p_n(q, \iota e)p_n(\iota e, q)$	$L\|n\|$
	Total	$(2L+1)\|n\| + 1$

□

Lemma 14.8. *Let* $e \cdot f \cdot g$ *be a combinatorial* 1*-cycle in* Δ. *Then* $\mathrm{Area}_{\mathcal{P}_H}(e^n f^n g^n) \leq 3|n|^2$.

Proof. Note that the relators efg and $e^{-1}f^{-1}g^{-1}$ imply that $ef = g^{-1} = fe$, so $[e, f]$ is null-homotopic with $\mathcal{P}_H$-Area 2. The following is a null $\mathcal{P}_H$-scheme for the word $e^n f^n g^n$:

j	σ_j	Area
1	$e^n f^n g^n$	$\|n\|$
2	$e^n f^n (f^{-1}e^{-1})^n$	$2\|n\|^2$
3	$e^n f^n f^{-n} e^{-n}$	0
	Total	$2\|n\|^2 + \|n\|$

□

Lemma 14.9. *Let* $e \cdot f \cdot g$ *be a combinatorial* 1*-cycle in* Δ. *Then* $\mathrm{Area}_{\mathcal{P}_H}\left(\Phi_n(efg)\right) \leq 3|n|^2 + (3L+6)|n| + 3$.

Proof. The following is a null $\mathcal{P}_H$-scheme for the word $\Phi_n(efg)$:

j	σ_j	Area
1	$p_n(q,\iota e)e^{n+1}p_n(\tau e,q)p_n(q,\iota f)f^{n+1}p_n(\tau f,q)\ldots$	
	$\ldots p_n(q,\iota g)g^{n+1}p_n(\tau g,q)$	$2L\|n\|$
2	$p_n(q,\iota e)e^{n+1}f^{n+1}g^{n+1}p_n(\tau g,q)$	$3\|n+1\|^2$
3	$p_n(q,\iota e)p_n(\tau g,q)$	$L\|n\|$
	Total	$3\|n\|^2+(3L+6)\|n\|+3$

□

Definition 14.10. Given a combinatorial 1-cycle C in Δ, a sequence $(C_i)_{i=0}^m$ of combinatorial 1-cycles is said to be *combinatorial null-homotopy* for C if $C_0 = C$, $C_m = \emptyset$ and each C_{i+1} is obtained from C_i by one of the following moves:

- 1-*cell expansion*: $C_i = e_1 \cdot \ldots \cdot e_l \rightsquigarrow C_{i+1} = e_1 \cdot \ldots \cdot e_k \cdot e \cdot \overline{e} \cdot e_{k+1} \cdot \ldots \cdot e_l$ for some k, where $e \in \mathrm{Edge}(\Delta)$;
- 1-*cell collapse*: Reverse of a 1-cell expansion;
- 2-*cell expansion*: $C_i = e_1 \cdot \ldots \cdot e_l \rightsquigarrow C_{i+1} = e_1 \cdot \ldots \cdot e_k \cdot e \cdot f \cdot g \cdot e_{k+1} \cdot \ldots \cdot e_l$ for some k, where $e \cdot f \cdot g$ is a combinatorial 1-cycle;
- 2-*cell collapse*: Reverse of a 2-cell expansion.

Lemma 14.11. *If $(C_i)_{i=0}^m$ is a combinatorial null-homotopy for the 1-cycle $e_1 \cdot \ldots \cdot e_l$ then the word $e_1^n \ldots e_l^n$ has $\mathcal{P}_H$-Area $\leq 3m|n|^2$.*

Proof. Given a combinatorial 1-cycle $C = e_1 \cdot \ldots \cdot e_l$, write $W_n(C)$ for the word $e_1^n \ldots e_l^n \in \mathrm{Edge}(\Delta)^{\pm *}$. If the 1-cycle C_i is obtained from C_{i-1} by a 1-cell expansion or collapse then, by repeated application of a relator $e\overline{e}$, the word $W_n(C_{i-1})$ can be converted to the word $W_n(C_i)$ at a $\mathcal{P}_H$-cost of at most $|n|$. If the 1-cycle C_i is obtained from C_{i-1} by a 2-cell expansion or collapse then, by lemma 14.8, the word $W_n(C_{i-1})$ can be converted to the word $W_n(C_i)$ at a $\mathcal{P}_H$-cost of at most $3|n|^2$.

Define m_1 to be the number of i for which C_i is obtained from C_{i-1} by a 1-cell expansion or collapse. Define m_2 to be the number of i for which C_i is obtained from C_{i-1} by a 2-cell expansion or collapse. Then the $\mathcal{P}_H$-Area of $e_1^n \ldots e_l^n = W_n(C)$ is at most $m_1|n| + 3m_2|n|^2 \leq 3(m_1+m_2)|n|^2 = 3m|n|^2$. □

Lemma 14.12. *There exists a constant K such that* $\mathrm{Area}_{\mathcal{P}_H}\left(p_n(q,\iota e)e^n p_n(\tau e,q)\right) \leq K|n|^2$ *for all $e \in \mathrm{Edge}(\Delta)$.*

Proof. Given $e \in \mathrm{Edge}(\Delta)$ write $\gamma_\iota(e)$ and $\gamma_\tau(e)$ respectively for the unique combinatorial geodesic paths in T from q to ιe and from τe to q. Then $\gamma_\iota(e) \cdot e \cdot \gamma_\tau(e)$ is a combinatorial 1-cycle for which there exists a combinatorial null-homotopy $\left(C_i(e)\right)_{i=0}^{m(e)}$ since Δ is simply-connected. By Lemma 14.11, $\mathrm{Area}_{\mathcal{P}_H}\left(p_n(q,\iota e)e^n p_n(\tau e,q)\right) \leq 3m(e)|n|^2$, so we can take $K = 3\max\{m(e) : e \in \mathrm{Edge}(\Delta)\}$. □

Lemma 14.13. *Let $e \cdot f \cdot g$ be a combinatorial 1-cycle in Δ. Then* $\mathrm{Area}_{\mathcal{P}_H}\left(\Phi_n(e^{-1}f^{-1}g^{-1})\right) \leq (3K+4)|n|^2 + (6L+6)|n| + 5$*, where K is the constant from Lemma 14.12.*

Proof. The following is a null $\mathcal{P}_H$-scheme for the word $\Phi_n(e^{-1}f^{-1}g^{-1})$:

j	σ_j	Area
1	$p_n(\tau e,q)^{-1}e^{-n-1}p_n(q,\iota e)^{-1}p_n(\tau f,q)^{-1}f^{-n-1}\dots$	
	$\dots p_n(q,\iota f)^{-1}p_n(\tau g,q)^{-1}g^{-n-1}p_n(q,\iota g)^{-1}$	$6L\lvert n\rvert$
2	$p_n(q,\tau e)e^{-n-1}p_n(\iota e,q)p_n(q,\tau f)f^{-n-1}p_n(\iota f,q)\dots$	
	$\dots p_n(q,\tau g)g^{-n-1}p_n(\iota g,q)$	0
3	$p_n(q,\iota f)e^{-n-1}p_n(\tau g,q)p_n(q,\iota g)f^{-n-1}p_n(\tau e,q)\dots$	
	$\dots p_n(q,\iota e)g^{-n-1}p_n(\tau f,q)p_n(q,\iota f)p_n(q,\iota f)^{-1}$	$3K\lvert n\rvert^2$
4	$p_n(q,\iota f)e^{-n-1}g^{-n}f^{-n-1}e^{-n}g^{-n-1}f^{-n}p_n(q,\iota f)^{-1}$	$2\lvert n\rvert+1$
5	$p_n(q,\iota f)e^{-n-1}(ef)^n f^{-n-1}e^{-n}(ef)^{n+1}f^{-n}p_n(q,\iota f)^{-1}$	$2\lvert n\rvert^2+2\lvert n+1\rvert^2$
6	$p_n(q,\iota f)e^{-n-1}e^n f^n f^{-n-1}e^{-n}e^{n+1}f^{n+1}f^{-n}p_n(q,\iota f)^{-1}$	0
7	$p_n(q,\iota f)e^{-1}f^{-1}efp_n(q,\iota f)^{-1}$	2
8	$p_n(q,\iota f)gg^{-1}p_n(q,\iota f)^{-1}$	0
	Total	$(3K+4)\lvert n\rvert^2$ $+(6L+6)\lvert n\rvert+5$

$\square$

Lemma 14.14. $\mathrm{Area}_{\mathcal{P}_H}\left(\Phi_{n+1}(e)\Phi_n(w_e)^{-1}\right) \leq 2K|n|^2+(3L^2+2L+2K)|n|+L+K$ *for all* $e\in\mathrm{Edge}(\Delta)$*, where* K *is the constant from Lemma 14.12.*

Proof. Note that if $e_1\cdot\ldots\cdot e_l$ is a combinatorial edge-path in Δ then $\Phi_n(e_1\dots e_l)=\prod_{i=1}^l p_n(q,\iota e_i)e_i^{n+1}p_n(\tau e_i,q)$ can be converted to $\prod_{i=1}^l p_n(q,\iota e_i)e_i^{n+1}p_n(q,\tau e_i)^{-1}\overset{\text{free}}{=} p_n(q,\iota e_1)e_1^{n+1}\dots e_l^{n+1}p_n(q,\tau e_l)^{-1}$ at a $\mathcal{P}_H$-cost of at most $lL|n|$. It follows that for all $u,v\in\mathrm{Vert}(\Delta)$ the word $\Phi_n\big(p(u,v)\big)$ can be converted to the word $p_n(q,u)p_{n+1}(u,v)p_n(q,v)^{-1}$ at a $\mathcal{P}_H$-cost of at most $L^2|n|$.

The following is a null $\mathcal{P}_H$-scheme for the word $\Phi_{n+1}(e)\Phi_n(w_e)^{-1}$:

j	σ_j	Area
1	$p_{n+1}(q,\iota e)e^{n+2}p_{n+1}(\tau e,q)\left[\Phi_n\big(p(q,\iota e)ep(\iota e,q)\big)\right]^{-1}$	$2L^2\lvert n\rvert$
2	$p_{n+1}(q,\iota e)e^{n+2}p_{n+1}(\tau e,q)\big[p_{n+1}(q,\iota e)p_n(q,\iota e)^{-1}\dots$	
	$\dots p_n(q,\iota e)e^{n+1}p_n(\tau e,q)p_n(q,\iota e)p_{n+1}(\iota e,q)\big]^{-1}$	0
3	$p_{n+1}(q,\iota e)e^{n+2}p_{n+1}(\tau e,q)p_{n+1}(\iota e,q)^{-1}\dots$	
	$\dots p_n(q,\iota e)^{-1}p_n(\tau e,q)^{-1}e^{-n-1}p_{n+1}(q,\iota e)^{-1}$	$L\lvert n+1\rvert$
4	$p_{n+1}(q,\iota e)e^{n+2}p_{n+1}(\tau e,q)p_{n+1}(q,\iota e)\dots$	
	$\dots p_n(q,\iota e)^{-1}p_n(\tau e,q)^{-1}e^{-n-1}p_{n+1}(q,\iota e)^{-1}$	$K\lvert n+1\rvert^2+K\lvert n\rvert^2$
5	$p_{n+1}(q,\iota e)e^{n+2}e^{-n-1}e^n e^{-n-1}p_{n+1}(q,\iota e)^{-1}$	0
	Total	$2K\lvert n\rvert^2$ $+(2L^2+L+2K)\lvert n\rvert$ $+L+K$

$\square$

Combining Lemmas 14.7, 14.9, 14.13 and 14.14 gives the following result.

Lemma 14.15. *The relational area function* RArea_H *of* $(\mathcal{P}_H^\infty,\|\cdot\|)$ *over* $\mathcal{P}_H$ *satisfies* $\mathrm{RArea}_H(l)\preceq l^2$.

Proof of Theorem 14.4. Since right-angled Artin groups are CAT(0) [21], A_Δ has some finite presentation which admits an area-radius pair (α, ρ) with $\alpha(l) \simeq l^2$ and $\rho(l) \simeq l$ [16, Proposition III.Γ.1.6.]. Thus, by Proposition 3.15, $\mathcal{P}'_A$ admits an area-radius pair (α', ρ') with $\alpha'(l) \simeq l^2$ and $\rho'(l) \simeq l$. By Theorem 7.4, (α', ρ') is an are-penetration pair for $(\mathcal{P}_H^\infty, \|\cdot\|)$ and hence, by Proposition 6.2 and Lemma 14.15, the Dehn function δ of $\mathcal{P}_H$ satisfies $\delta(l) \preceq l^4$. □

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